LECTURE 17: DERIVATION OF WAVE EQUATION AND CONCEPTS OF PDES

From the purely mathematical point of view, a PDE is an equation that contains one or more partial derivatives of an unknown function that depends on at least two variables. PDEs are very important in dynamics, elasticity, heat transfer, electromagnetic theory, and quantum mechanics. They have a much wide range of applications. Thus PDEs are subjects of many ongoing research and development projects.

In this lecture, we realize some important PDEs arising in physics, and derive wave equation of the vibrating elastic string.

1. Vibrating String Model: Wave Equation

In this section we derive the equation of the vibrating string. This equation is the one-dimensional form of the wave equation, which occurs throughout many branches mathematical physics.

1. Physical phenomena

We state the Physical phenomena as follows: an flexible elastic string, such as a violin string, vibrates transversely a small displacement in a plane.

For convenience we take a system of rectangular coordinates such that $u$-axis is the direction of motion of the string and $x$-axis is equilibrium position of the string. We place the string along the $x$-axis, stretch it to length $L$, then fasten it at the ends $x = 0$ and $x = L$. We then distort the string, and at some instant, call it $t = 0$, we release it and allow it to vibrate. The problem is to determine the vibrations of the string, that is, to find its deflection $u(x, t)$ at any point $x$ and at any time $t > 0$, see Figure 1 as follows.

![Figure 1. Deflected string at fixed time t](image)

Physical Assumptions

1. The mass of the string per unit length is constant ("homogeneous string"). The string is perfectly elastic and does not offer any resistance to bending.

2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.

3. The string performs small transverse motions freely in a vertical plane, which implies that there is no external force exerted this string and every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.
Under these assumptions we may expect solutions $u(x, t)$ that describe the physical reality sufficiently well.

2. Derivation of the PDE of the Model ("Wave Equation")

To obtain the equation of vibrating string, we consider the forces acting on a small portion of the string (see Figure 1). This method is typical of modeling in mechanics and elsewhere.

Since the string offers no resistance to bending, the tension is tangential to the curve of the string at each point. Let $T_1$ and $T_2$ be the tension at the endpoints $P$ and $Q$ of that portion. Since the points of the string move vertically, there is no motion in the horizontal direction. Hence the horizontal components of the tension must be constant. Using the notation shown in Figure 1, we thus obtain

$$T_1 \cos \alpha = T_2 \cos \beta = T = \text{const.} \tag{1.1}$$

Here we write $T_i = |T_i|$ for $i = 1, 2$.

Since the string vibrate freely in the vertical direction, there are only two forces, namely, the vertical components $-T_1 \sin \alpha$ of $T_1$ at $P$ and $T_2 \sin \beta$ of $T_2$ at $Q$. Here the minus sign appears because the component at $P$ points to the opposite direction of $u$ and $\sin \alpha$ is positive. By Newton's second law the resultant of these two forces is equal to the mass $\rho \Delta x$ of the portion times the acceleration $\frac{\partial^2 u}{\partial t^2}$ evaluated at some point between $x$ and $x + \Delta x$, where $\rho$ is the mass of the undeflected string per unit length and $\Delta x$ is the length of the portion of the undeflected string. ($\Delta$ is generally used to denote small quantities. It has nothing to do with the Laplacian $\nabla^2$, which is sometimes also denoted by $\Delta$.) Hence

$$T_2 \sin \beta - T_1 \sin \alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}. \tag{1.2}$$

Using (1.1) and dividing the above by $T_2 \cos \beta = T_1 \cos \alpha = T$, we obtain

$$\frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} = \tan \beta - \tan \alpha = \frac{\rho \Delta x \frac{\partial^2 u}{\partial t^2}}{T}. \tag{1.2}$$

Now $\tan \alpha$ and $\tan \beta$ are the slopes of the string at $x$ and $x + \Delta x$, i.e.,

$$\tan \alpha = \left( \frac{\partial u}{\partial x} \right)_x, \quad \tan \beta = \left( \frac{\partial u}{\partial x} \right)_{(x+\Delta x)}. \tag{1.2}$$

Here we have to write partial derivatives because $u$ also depends on time $t$. Dividing (1.2) by $\Delta x$, we thus have

$$\frac{1}{\Delta x} \left[ \left( \frac{\partial u}{\partial x} \right)_{(x+\Delta x)} - \left( \frac{\partial u}{\partial x} \right)_x \right] = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}. \tag{1.2}$$

Letting $\Delta x \to 0$, we obtain

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ t > 0, \tag{1.3}$$

where $c^2 = \frac{T}{\rho}$. This is called the one-dimensional wave equation. The physical constant $\frac{T}{\rho}$ is denoted by $c^2$ (instead of $c$) to indicate that this constant is positive, a fact that will be essential to the form of the solutions. "One-dimensional" means that the equation involves only one space variable $x$.

If the string is subject to an external force along vertical direction, say $h_0(x, t)$ per unit length at $x$ and time instant $t$, then by a similar way we can derive the equation of small transverse vibrating string is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + h(x, t), \quad 0 < x < L, \ t > 0, \tag{1.4}$$

where $h(x, t) = Th_0(x, t)/\rho$. It is also one-dimensional wave equation. By comparison with (1.3), wave equation (1.4) indicates that the vibrating string is subject to an external.
We see that equations (1.3) and (1.4) model the vibrating string. They also occur many physical problems such as electromagnetic theory and plasma physics.

2. Basic Concepts of PDEs

1. Basic Concepts of PDEs

From equations of the vibrating string (1.3), (1.4) in previous part, we see that there exist one or more partial derivatives in both (1.3) and (1.4).

In general, a partial differential equation (PDE) is an equation involving one or more partial derivatives of an (unknown) function, call it $u$, that depends on two or more variables, often time $t$ and one or several variables in space. The order of the highest derivative is called the order of the PDE. Just as was the case for ordinary differential equations (ODEs), second-order PDEs will be the most important ones in applications.

We say that a PDE is linear if it is of the first degree in the unknown function $u$ and its partial derivatives. Otherwise we call it nonlinear. We call a PDE homogeneous if each of its terms contains either $u$ or one of its partial derivatives. Otherwise we call the equation nonhomogeneous.

Thus, both (1.3) and (1.4) are second-order linear PDEs. Since in general $h(x, t)$ is not identically zero, (1.4) is nonhomogeneous equations, whereas (1.3) is homogeneous. Otherwise we call the equation nonhomogeneous.

Example 1. Apart from (1.3) and (1.4) in previous part, we will realize many PDEs in subsequent lectures. Here we give some of them in which $c$ is a positive constant, $t$ is time, $x, y, z$ are Cartesian coordinates, and dimension is the number of these coordinates in the equation.

(1) Equations

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + h(x, t).$$

It is one-dimensional heat equation modeling a rod that is capable of conducting heat, and in which $u(x, t)$ represents the temperature at the position $x$ at time instant $t$, and $h(x, t)$ represents that there exists a heat source in rod. When $h(x, t) \equiv 0$, it is a homogeneous equation, otherwise nonhomogeneous.

(2) Two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

which models a small transverse motion of a membrane.

(3) Two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ is a three-dimensional Laplace equation, whereas $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ is a two-dimensional Poisson equation. They all occur electromagnetic theory and electrostatic potentials theory.

The above equations are all second-order linear PDEs. There are many nonlinear PDEs and higher order PDEs in physics, such as Schrödinger equation

$$i \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + |u|^2 u = 0,$$
which is also a complex PDE, and Korteweg-de Vries (KdV) equation
\[
\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.
\]

A solution of a PDE in some region \( R \) of the space of the independent variables is a function that has all the partial derivatives appearing in the PDE in some domain \( D \) containing \( R \), and satisfies the PDE everywhere in \( R \).

Often one merely requires that the function is continuous on the boundary of \( R \), has those derivatives in the interior of \( R \), and satisfies the PDE in the interior of \( R \). Letting \( R \) lie in \( D \) simplifies the situation regarding derivatives on the boundary of \( R \), which is then the same on the boundary as it is in the interior of \( R \).

We know that if an ODE is linear and homogeneous, then from known solutions we can obtain further solutions by superposition. For PDEs the situation is quite similar.

**Theorem 2.1.** (Superposition principle) If \( u_1 \) and \( u_2 \) are solutions of a homogeneous linear PDE in some region \( R \), then
\[
u = c_1 u_1 + c_2 u_2
\]
with any constants \( c_1 \) and \( c_2 \) is also a solution of that PDE in the region \( R \).

Since the proof of this theorem is quite similar to that of ODE, we omit it.

**Example 2.** Find solutions \( u(x, y) \) of the PDE \( u_{xx} - u = 0 \) depending on \( x \) and \( y \).

**Solution.** Since no \( y \)-derivatives occur, we can solve this PDE like ODE \( v'' - v = 0 \), which has general solution \( v(x) = Ae^x + Be^{-x} \) with constant \( A \) and \( B \). When \( A \) and \( B \) are functions of \( y \), this representation also satisfies the PDE. Thus we obtain the solution
\[
u(x, y) = A(y)e^x + B(y)e^{-x}
\]
with arbitrary twice differentiable functions \( A(y) \) and \( B(y) \). We thus have a great variety of solutions.

**Example 3.** Find solutions \( u = u(x, y) \) of this PDE \( u_{xy} = -u_x \).

**Solution.** Setting \( u_x = p \), we have \( p_y = -p \). Just as Example 2, we solve it like an ODE and replace the constant by an arbitrary function. Thus, from \( \frac{dp}{dx} = -1 \) we obtain \( \ln |p| = -y + \tilde{c}(x) \), \( p = c(x)e^{-y} \). Integration with respect to \( x \) gives the solution
\[
u(x, y) = f(x)e^{-y} + g(y),
\]
where \( f(x) = \int c(x)dx \) and \( g(y) \) are arbitrary differentiable functions.

**Remark.** The method in Examples 2 and 3 only can be used for some special PDEs that involve derivatives with respect to one variable only or can be transformed to such a form.

2. **Additional conditions for PDEs**

In general, the totality of solutions of a PDE is very large. For example, the functions
\[
u = x^2 - y^2, \quad u = e^x \cos y, \quad u = x \sin x \cosh y, \quad u = \ln(x^2 + y^2)
\]
which are entirely different from each other, are solutions of two-dimensional Laplace equation in Example 1.

In order specify a solution of a PDE corresponding to a given physical problem, we have to use “additional conditions” arising from the problem. These additional conditions often consist of boundary conditions and initial conditions.

When time \( t \) is one of the variables, initial conditions prescribe the value of solution \( u \) (or both \( u_t = \frac{\partial u}{\partial t} \) and \( u \) according to the highest order derivative with respect to \( t \)) at \( t = 0 \). Boundary
condition signifies that the solution \( u \) (or some of its derivatives) is evaluated on the boundary of the region \( R \).

For one-dimensional wave equations (1.3) and (1.4) in which \( 0 < x < L \), there are two initial conditions that are represented as

\[
\begin{align*}
    u(x, 0) &= f(x), \quad 0 < x < L, \\
    u_t(x, 0) &= g(x), \quad 0 < x < L,
\end{align*}
\]

since the highest order derivative with respect to \( t \) in equations (1.3) and (1.4) is two. Initial condition (2.1) indicates the displacement of the string is known at time \( t = 0 \) and is given by a function \( f(x), \quad 0 < x < L \). Initial condition (2.2) signifies the velocity at \( t = 0 \) is given by a function \( g(x), \quad 0 < x < L \).

Often there exist three types of boundary conditions at the endpoint \( x = 0 \):

\[
\begin{align*}
    (I) & \quad u(0, t) = h_1(t), \quad t \geq 0, \\
    (II) & \quad -u_x(0, t) = h_2(t), \quad t \geq 0, \\
    (III) & \quad -u_x(0, t) + ku(0, t) = 0, \quad t \geq 0.
\end{align*}
\]

Here \( k > 0 \). Boundary condition (I) signifies that the displacement of the string at the end \( x = 0 \) is known and is given by a function \( h_1(t), \quad t \geq 0 \), which is called Dirichlet boundary condition or the first boundary value condition. For instance, if the end is fixed, then we can write this boundary condition \( u(0, t) = 0, \quad t \geq 0 \). Boundary condition (II), is called Neumann boundary condition or the second boundary value condition, signifies that the end \( x = 0 \) is subject to an external force \( h_2(t), \quad t \geq 0 \). If no external force is present at \( x = 0 \) which implies at this point the string can move freely, we write \( u_x(0, t) = 0 \). Boundary condition (III) indicates the end \( x = 0 \) is fastened at the spring with a coefficient \( k \), which is mixed boundary condition or the third boundary value condition. In general, boundary condition (III) can be written as \( -u_x(0, t) + ku(0, t) = h_3(t), \quad t \geq 0 \).

Similarly, we can have each of the three boundary conditions present at the end \( x = L \), namely,

\[
\begin{align*}
    (I) & \quad u(L, t) = h_1(t), \quad t \geq 0, \\
    (II) & \quad u_x(L, t) = h_2(t), \quad t \geq 0, \\
    (III) & \quad u_x(L, t) + k_1 u(L, t) = 0, \quad t \geq 0,
\end{align*}
\]

where \( k_1 > 0 \). The signification of the functions \( h_1(t) \) and \( h_2(t) \) are the same as for the endpoint \( x = 0 \), the representations may be different from those in \( x = 0 \).
3. Problems

1. Derive the equation of small transverse motion of the string that is subject to an external force that is proportional to velocity, i.e., \( f(x,t) = -k u_t(x,t) \). For simplicity, the string is placed in \( x \)-axis at \([0,L]\).

2. Verify that \( u(x,t) = \nu(x + ct) + \omega(x - ct) \) with any twice differentiable functions \( \nu \) and \( \omega \) satisfies (1.3).

3. Verify that \( u = \sin kct \cos kx \) with real number \( k \) is a solution of (1.3).

4. Verify that \( u = e^{(-\omega^2c^2t)} \cos \omega x \) with real number \( \omega \) is a solution of homogeneous heat equation in Example 1 (1).

5. Verify that the function \( u(x, y) = a \ln(x^2+y^2)+b \) satisfies two-dimensional Laplace's equation. Find \( a \) and \( b \) so that \( u \) satisfies the boundary conditions \( u = 110 \) on the circle \( x^2 + y^2 = 1 \) and \( u = 0 \) on the circle \( x^2 + y^2 = 100 \).

6. Solve \( u = u(x,y) \) for the following PDEs
   
   (1) \( u_{xx} + 16\pi^2 u = 0. \)
   
   (2) \( 25u_{yy} - 4u = 0. \)
   
   (3) \( u_y + y^2 u = 0. \)
   
   (4) \( u_{yy} + 6u_y + 13u = 4e^3y. \)
   
   (5) \( u_{xy} = u_x. \)
   
   (6) \( x^2 u_{xx} + 2xu_x - 2u = 0. \)