HW1, due to 10/17/2021

Exercise 1. Given a metric space (X, d) we define a relation $\sim$ on X by

$$x \sim y \iff d(x, y) < \infty.$$

Then $\sim$ is an equivalence relation on X, and the equivalence class $[x]$ of x can be endowed with a natural metric (still denoted by d). For any $x \in X$, show that $([x], d)$ is a finite metric space.

Exercise 2. Given a semi-metric space (X, d) we define a relation $\sim$ on X by

$$x \sim y \iff d(x, y) = 0.$$

Show that $\sim$ is an equivalence relation on X. Define

$$\hat{X} := X/\sim = \{[x] : x \in X\}, \quad \hat{d}([x], [y]) := d(x, y).$$

Show that $(X/d, d) := (\hat{X}, \hat{d})$ is a metric space.

Exercise 3. Let $X = \mathbf{R}^2$ and define

$$d((x, y), (x', y')) := |(x - x') + (y - y')|.$$

Show that d is a semi-metric on X. Define $\boldsymbol{f} : \mathbf{R}^2/d \to \mathbf{R}$ by $\boldsymbol{f}([(x, y)]) := x + y$. Show that $\boldsymbol{f}$ is an isometry.

Exercise 4. Consider the set X of all continuous real-valued functions on $[0, 1]$. Show that

$$d(f, g) := \int_0^1 |f(x) - g(x)| dx$$

defines a metric on X. Is this still the case if continuity is weakened to integrability?

Exercise 5. Let (X, d_X) and (Y, d_Y) be two metric spaces. We define

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) := \sqrt{d_X(x_1, x_2) + d_Y(y_1, y_2)}.$$

Show that $(X \times Y, d_{X\times Y})$ is a metric space.

Exercise 6 (Bonus). As a subspace of $\mathbf{R}^2$, the unit circle $\mathbf{S}^1$ carries the restricted Euclidean metric from $\mathbf{R}^2$. We can define another (intrinsic) metric d_{int} by

$$d_{\text{int}}(\boldsymbol{x}, \boldsymbol{y}) := \text{the length of the shorter arc between them.}$$

Note that $d_{\text{int}}(\boldsymbol{x}, \boldsymbol{y}) \in [0, \pi]$ for any points $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{S}^1$.

(a) Show that any circle arc of length less than or equal to π, equipped with the intrinsic metric, is isometric to a straight line segment.

(b) The whole circle with the intrinsic metric is not isometric to any subset of $\mathbf{R}^2$.