Coordinative Control of Multi-agent Networks with Switching Topology and Delayed Communications

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Background: Collective behavior

- Group movement of creatures, e.g. a flock of birds
- Coordinative control in man-made systems, e.g. formation of a group of unmanned aerial vehicles
- Agreement in social behavior, e.g. people’s opinions in voting
Basic Observations

All individuals behave uniformly (e.g. the same velocity, position, opinion), which may be affected by

1. Each Individual’s (called agent) behavior
2. Interactions with others

Question:
How can they achieve collective behaviors?
Two Typical Problems

- **Leader following**: A set of agents is requested to track a specified agent.
  - The leader has its own decided motion (not affected by the others)
  - Some followers can receive the leader’s information
  - Followers interact with each other in certain ways
  - Under what conditions, all the followers will track the leader?

- **Consensus (without leader)**
  - The agents interact, under what conditions they can reach uniformly behavior by themselves?
Modeling of Individual Agent

It is assumed that, the agent is governed by the following differential equation

\[ \dot{x} = f(x) \]

\(x\) stands for the properties interested such as position, velocity et al.

Note:
1. \(f(x)\) may be nonlinear e.g. Chua’s chaotic oscillator.
2. In many studies, \(f(x)\) may be simplified as linear function.

\[
\begin{aligned}
C_1 \frac{dV_1}{dt} &= \frac{1}{R} (V_2 - V_1) - f(V_1) \\
C_2 \frac{dV_2}{dt} &= \frac{1}{R} (V_1 - V_2) + i_L \\
L \frac{di_L}{dt} &= -V_2 - R_0 i_L
\end{aligned}
\]

Nonlinear chaotic oscillator
Modeling of Interactions

- Mathematical Representation
  - Who affects who? *(Adjacency matrix of a graph)*
    
    \[
    A = (a_{ij})
    \]
    
    \[
    a_{ij} = \begin{cases} 
    1, & \text{if information } j \to i \ (i \neq j) \\
    0, & \text{otherwise}
    \end{cases}
    \]

  - How strong is the influence?
    A positive number may be assigned to each link.

  - The deviations between agents are usually used to drive an agent (it is similar to error feedback in control theory).

  **E.g. For agent 1**

  \[
  \dot{x}_1 = f(x_1) + 3 \cdot (x_2 - x_1) + 2 \cdot (x_3 - x_1) \\
  = f(x_1) + \sum_{j=1}^{4} \overline{a}_{ij} (x_j - x_1) = f(x_1) + \sum_{j=1}^{4} g_{ij} x_j
  \]

  **Note:** For mathematical manipulations, let

  \[
  G = (g_{ij}), \quad g_{ij} = \begin{cases} 
  \overline{a}_{ij}, & i \neq j \\
  - \sum_{k=1, k \neq i}^{N} \overline{a}_{ik}, & i = j
  \end{cases}
  \]
Interactions Between Agents

Different settings w.r.t. the problem focused:

- **Undirected network**: Bidirectional communication between agents is available, e.g. oscillators coupled via springs.
- **Directed network**: The direction of communications between agents is specified.
- **Communication delayed**: The communication delay is considered.
- **Switched topology**: The topology changes intermittently, e.g. mobile sensors communications affected by distances / obstacles / hardware fault.
Current Studies

- Usually, the agent dynamics are simplified as integrator or linear dynamics with low dimension.
  
  **BUT …** nonlinear dynamics widely exist, e.g. chaotic oscillator or nonlinear model of robots.

- For leader following
  
  - The communication is assumed to be fixed.
    
    **BUT …** In practice, it may change with time and the leader is not always available for the followers

- For agent consensus
  
  - For nonlinear agents, the communications are assumed to be fixed.
    
    **BUT …** The topology of the network may vary with time. In some cases, some of agents may even be isolated. Under this condition, how to ensure consensus?
Main Contributions

Assuming that all agents have nonlinear dynamics, and the communication delay exists,

Criteria are established for

• Achieving successful leader following when the followers have connected but switched topology

• Achieving consensus for a time-varying directed network of agents

• Achieving successful leader following for switching directed graph

• Achieving successful consensus for fast switching topology
1st Study: Leader Following for Switching Connective Topology

Leader: \( \dot{x}_0 = Ax_0 + f(x_0) \)

Followers with the leader’s Information:
\[
\dot{x}_i = Ax_i + f(x_i) + \sum_{j=1}^{N} g^{\sigma(i)}_{ij} x_j(t - \tau(t)) - b_i (x_i - x_0)
\]

Followers without the leader’s Information:
\[
\dot{x}_i = Ax_i + f(x_i) + \sum_{j=1}^{N} g^{\sigma(i)}_{ij} x_j(t - \tau(t))
\]

\( f(x) \): dynamics of an isolated agent
\( \tau(t) \): communication delay between agents
\( \sigma(t) : [0, \infty) \rightarrow \{1,2,\cdots,m\} \)
\( G_{\sigma(t)} = \left( g^{\sigma(t)}_{ij} \right) \): network matrix at time \( t \)
\( b_i \) indicates if agent \( i \) knows the leader
\( B = \text{diag}(b), b = (b_1, b_2, \cdots, b_N) \)

Links between followers may switch

\[
G_1 = \begin{pmatrix}
-3 & 1 & 1 & 1 \\
1 & -3 & 1 & 1 \\
1 & 1 & -3 & 1 \\
1 & 1 & 1 & -3
\end{pmatrix};
G_2 = \begin{pmatrix}
-2 & 1 & 0 & 1 \\
1 & -2 & 0 & 1 \\
0 & 0 & -1 & 1 \\
1 & 1 & 1 & -3
\end{pmatrix};
\]

\[
G_3 = \begin{pmatrix}
-1 & 0 & 0 & 1 \\
0 & -1 & 0 & 1 \\
0 & 0 & -1 & 1 \\
1 & 1 & 1 & -3
\end{pmatrix};
B = \begin{pmatrix}
1 \\
1 \\
1 \\
0
\end{pmatrix}.
\]
Problem Analysis

Defining \( e_i = x_i - x_0 \) as the tracking error of i-th agent, and

\[
e(t) = (e'_1 \quad e'_2 \quad \ldots \quad e'_N)
\]

the tracking error satisfies the following dynamics

\[
\dot{e}(t) = I_N \otimes C e(t) + F(x) - F(x_0) + G_{\sigma(t)} \otimes I_n e(t - \tau(t)) - B \otimes I_n e(t)
\]

with \( B = \text{diag}(b_1, b_2, \ldots, b_N) \), \( F(x_0) = I_{N \times 1} \tilde{f}(x_0) \), \( F(x) = (\tilde{f}(x_2)', \tilde{f}(x_2)', \ldots, \tilde{f}(x_2)')' \)

To ensure successful leader following, exponential stability of the error system is established by using Lyapunov functional theory. One can select

\[
V(t) = V_1(t) + V_2(t) + V_3(t)
\]

with

\[
V_1(t) = e(t)'(I_N \otimes P)e(t);
\]

\[
V_2(t) = \int_{t-\tau(t)}^{t} e(s)'(I_N \otimes Q_1)e(s)ds + \int_{t-\tau(t)}^{t} e(s)'(I_N \otimes Q_2)e(s)ds;
\]

\[
V_3(t) = \int_{-h}^{0} \int_{t+\theta}^{t} \dot{e}'(\varpi)(I_N \otimes R)e(\varpi)d\varpi d\theta.
\]
Theoretical Results

- **Theorem 1:** The followers will asymptotically follow the leader if
  - There exist some positive matrices $P$, $Q$, and $R$ such that a specific set of linear matrices inequalities can be satisfied.
- Proof based on common Lyapunov functional
  - $\dot{V}(t) \leq W(t) < 0$
    - $W(t)$ independent on the switch.
  - Some LMI conditions are obtained
    - (Please refer to IEEE TCAS-I 2011 No.11)
- Applicable for arbitrary switching.
- Remark: Solution of LMIs is readily solved by using MATLAB.
Two Corollaries

- **Corollary 1:** The followers can track the leader exponentially if there exist some positive matrix $P$, and a diagonal positive matrix $\Sigma > 0$ such that

$$PA + A'P + 2\bar{\lambda}_{H_k}P + l^2\Sigma - P\Sigma^{-1}P < 0$$

With $\bar{\lambda}_{H_k} = \max_i(\lambda_{k,i})$, $\lambda_{k,i}$ stands for the largest eigenvalues of $G_k - B$.

- **Corollary 2:** The followers can track the leader exponentially if

$$\bar{\lambda}_{C+C'} + l^2 < 1 - 2\bar{\lambda}_{H_k}$$

where $l > 0$ indicates the Lipschitz constant.
Tracking under Different Switching Signals

Five modified Chua’s oscillators: 4 followers and 1 leader.

**Periodically switch:**
Switches by following the order 1, 2, 3, repeatedly, while each topology lasts for 0.2 second.

**Randomly switch:**
Switches in a random order with the duration also randomly assigned in the range of (0, 0.1).

**Note:** Tracking error: \( E(t) = \sum_{i=1}^{N} ||x_i - x_0|| \)
Agent with interactions:

\[ \dot{x}_i(t) = Ax_i + f(x_i) + \sum_{j=1}^{N} g_{ij}^{\sigma(t)} x_j(t - \tau(t)) \]

\[ i = 1, 2, \ldots, N \]

\( G_{\sigma(t)} = \left( g_{ij}^{\sigma(t)} \right) \) stands for a directed network (also called a digraph) appearing at time \( t \).

\( \sigma(t) : [0, \infty) \rightarrow \{1, 2, \ldots, m\} \)

denotes the switching signal for topology, and \( m \) is totally number of topologies.

**Objective:** Find the conditions that all agents can reach consensus.
Problem Analysis

- Considering the synchronization manifold as follows

$$\Delta = \{x : x_i = x_j, \forall 1 \leq i, j \leq N\}$$

which belongs to the sub-space

$$\text{span}\{I_N \otimes z\}, \forall z \in \mathbb{R}^n$$

For strongly connected digraph, $U = I_N - (1_{N \times N} / N)$ leads to the distance defined as

$$D(x, \Delta) = \sqrt{x'U \otimes I_n x}$$

which may be regarded as the synchronization error.

Selecting the Lyapunov functional as $V_k(t) = V_{k,1}(t) + V_{k,2}(t) + V_{k,3}(t) + V_{k,4}(t)$, where

$$V_{k,1}(t) = x(t)'U \otimes P_k x(t)$$

$$V_{k,2}(t) = \int_{t-h}^t e^{\alpha(t-s)} x(s)'U \otimes Q_{1k} x(s) \, ds$$

$$V_{k,3}(t) = \int_{t-h}^t e^{\alpha(t-s)} x(s)'U \otimes Q_{2k} x(s) \, ds$$

$$V_{k,4}(t) = \int_{-h}^0 \int_{t+\theta}^{t-h} e^{\alpha(t-s)} x(s)'U \otimes (R_{1k} + R_{2k}) x(s) \, ds \, d\theta$$

$V(t) \to 0 \Leftrightarrow D(x, \Delta) \to 0$
Theoretical Results

**Theorem 2:** Given a finite digraph set $\Theta = \{G_1, G_2, \ldots, G_m\}$, all agents will reach consensus exponentially if

1. The exist a constant $\mu \geq 1$, and, for each $G_i \in \Theta$, there exist a constant $\alpha_i$ and positive $P, Q_1, Q_2, R_1, R_2$ satisfying a set of linear matrix inequalities.

**Note:** $\alpha_i < 0$ implying $G_i$ ensures agents' consensus, otherwise it doesn't.

2. Given a specified $\alpha^* < 0$, partition $\Theta$ into $\Theta_1, \Theta_2$ with

$$\begin{cases} 
\alpha_k < \alpha^*, \forall G_k \in \Theta_1; \\
\alpha^* \leq \alpha_k, \forall G_k \in \Theta_2.
\end{cases}$$

$$\begin{align*}
\alpha^- &= \max \{\alpha_k, G_k \in \Theta_1\} \\
\alpha^+ &= \max \{\alpha_k, G_k \in \Theta_2\}
\end{align*}$$

- $F$ : Switch frequency inside each interval;
- $F^+$ : Switch frequency from $\Theta_2$ and $\Theta_1$;
- $T^-$ : Total time for $\Theta_1$;
- $T^+$ : Total time for $\Theta_2$.

$T^- = T^+$

$\alpha = (0,-\alpha^*)$.

Note: The LMI conditions can be found in p.55 of the thesis.
Example: Two Digraphs

Given a digraph set \( \Theta = \{ G_1, G_2 \} \), all agents will reach consensus exponentially if

- The exist a constant \( \mu \geq 1 \), and, for each \( G_i \in \Theta \), there exist a constant \( \alpha_i \) such that a set of linear matrix inequalities can be satisfied. Assume that \( \alpha_1 < 0, \alpha_2 > 0 \).

- Given a specified \( \alpha^* \in [\alpha_1, 0) \), the following conditions hold

\[
T^- / T^+ \geq (\alpha^+ - \alpha^* + 2F^+) / (\alpha^* - \alpha^- - 2F^+);
\]
\[
F^+ \leq \alpha / (h(\alpha^+ - \alpha^-)), \quad \alpha \in (0, -\alpha^*).
\]

in which

- \( T^- \): Total time for \( G_1 \);
- \( T^+ \): Total time for \( G_2 \);
- \( F^+ \): Switch frequency.
Numerical Simulations

5 agents with modified Chua’s chaotic dynamics

Switching rule obtained: \( \frac{T^-}{T} \geq 0.46, F^+ \leq 1. \)

Note: the coupling strength is 3.
Numerical Simulations

Consensus error for non-periodic switching signal for 5 agents

Consensus error for 100 agents with 3 possible random topologies
3rd Study: Leader Following over Switching Digraph

Leader: \( \dot{x}_0 = f(x_0) \)

Followers with the leader’s Information:
\[
\dot{x}_i = f(x_i) + c \sum_{j=1}^{N} g_{ij}^{\sigma(t)} x_j (t - \tau(t)) - b_i(x_i(t - \tau(t)) - x_0(t - \tau(t)))
\]

Followers without the leader’s Information:
\[
\dot{x}_i = f(x_i) + c \sum_{j=1}^{N} g_{ij}^{\sigma(t)} x_j (t - \tau(t))
\]

\( c > 0 \) indicates the coupling strength
\( B = \text{diag}(b), b = (b_1, b_2, \cdots, b_N) \)
Theoretical Results

- For Case I: all possible topologies can ensure leader following

- **Theorem 3** For the leader following model with delay $\tau(t) \leq h$, if there exists a positive constant $\alpha > 0$ and a series of positive matrices such that the LMI condition holds, all the followers can track the leader successfully for the following average dwell time

$$T_a > \frac{\ln \mu}{\alpha}$$

with

$$P_k \leq \mu \frac{\min(\zeta_j)}{\max(\zeta_k)} P_j, \quad R_k \leq \mu \frac{\min(\zeta_j)}{\max(\zeta_k)} R_j, \quad k, j = 1, 2, \ldots, m, \text{ and } \mu \geq 1.$$
Numerical Simulations

- **Case I:**
  - Each agent is Chua’s chaotic oscillator
  - Switching condition obtained

\[ T_a > 0.4714 \]

**Topology 1:**

\[
G_1 = \begin{pmatrix}
-3 & 1 & 1 & 1 \\
0 & -2 & 1 & 1 \\
1 & 0 & -2 & 1 \\
1 & 0 & 1 & -2
\end{pmatrix},
B_1 = \begin{pmatrix}
-1 \\
0 \\
-1
\end{pmatrix}
\]

**Topology 2:**

\[
G_2 = \begin{pmatrix}
-2 & 1 & 0 & 1 \\
1 & -2 & 0 & 1 \\
1 & 1 & -3 & 1 \\
1 & 1 & 0 & -2
\end{pmatrix},
B_2 = \begin{pmatrix}
-1 \\
-1 \\
0
\end{pmatrix}
\]
Theoretical Results

- For Case II: Not all topologies can ensure leader following

- Theorem 4  For the leader following model with delay \( \tau(t) \leq h \), if there exist some positive constants \( \alpha_k > 0 \) and some matrices such that a set of LMIs hold, all followers can track the leader for any switching signal satisfying

\[
\begin{align*}
F & \leq \frac{\alpha - \alpha^*}{\ln \mu} \\
T^- & \geq \frac{\alpha^* - \alpha^- + F \ln \mu}{\alpha^+ - \alpha^* - F \ln \mu}
\end{align*}
\]

where \( \alpha \in (0, \alpha^+), \alpha^* \in (\alpha, \alpha^+) \) and \( F \) stands for the switching frequency which is defined as \( F(0,t) = N(0,t)/t \).

**Note:** \( \alpha^+ = \max_{\Theta_k \in \Theta_1} (\alpha_k), \alpha^- = \min_{\Theta_k \in \Theta_2} (\alpha_k) \) where the topology set ensures leader following and fails to ensure leader following are indicated as \( \Theta_1, \Theta_2 \).

Please refer to IEEE TCAS-I 2012 (12)
Numerical Simulations

- **Case II:**
  - The same agent model
  - Switching condition obtained

\[ F \leq 1, \frac{T^-}{T^+} \geq 2.5 \]

**Topology 1:**
Defined as in case I

**Topology 2:**

\[
G_2 = \begin{pmatrix}
-2 & 1 & 1 & 0 \\
1 & -1 & 0 & 0 \\
1 & 1 & -2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
B_2 = \begin{pmatrix}
-1 \\
-1 \\
0 \\
0
\end{pmatrix}
\]

Note: In Topology 2, 4\textsuperscript{th} agent cannot reach the leader.
Theoretical Results

- For Case III: leader following with switching disconnected topology

- **Theorem 5** Given a leader-follower network model satisfies for each strongly connected component, if

\[
\alpha_1 \Delta t_i - \alpha_2 (T - \Delta t_i) > (m - 1) \ln \mu
\]

holds, then all the followers can track the leader exponentially.

- **Corollary 3** When there are only two possible networks, successful leader following is achieved if

\[
\max \left\{ \frac{\ln \mu}{\lambda_1 - \lambda_2}, \frac{\lambda_0 \Delta t_j + \ln \mu}{\lambda_1} \right\} \leq \Delta t_i \leq \frac{\lambda_1 \Delta t_j - \ln \mu}{\lambda_0}, (i, j = 1, 2, i \neq j).
\]
Numerical Simulations

- Case III:
  - The same agent model
  - Switching condition obtained

\[
\frac{9.4 \Delta t_j + \ln 1.1}{12.7} \leq \Delta t_i \leq \frac{12.7 \Delta t_j - \ln 1.1}{9.4} \quad (i, j = 1, 2).
\]

### Topology 1:
\[
\begin{align*}
G_1 &= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \\
B_1 &= \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}
\end{align*}
\]

### Topology 2:
\[
\begin{align*}
G_2 &= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
B_2 &= \begin{pmatrix} 0 \\ 0 \\ -5 \\ 0 \end{pmatrix}
\end{align*}
\]

Note: In Topology 1, 3rd, 4th agents cannot reach the leader while in Topology 2, 1st, 2nd agents cannot reach the leader.
Network model considered

\[ \dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^{N} g_{ij}(t)x_j(t - \tau(t)) \]

Fast switching network model

\[ \dot{x}(t) = f(x) + c(G(t/\epsilon) \otimes I_n)x(t - \tau) \]

satisfying the averaging condition

\[ \frac{1}{T} \int_{t}^{t+T} G(w)dw = \bar{G} \]

Objective: find conditions ensuring consensus when the network undergoes fast switching.
Theoretical Results

- **Theorem 6** If the time-varying network satisfies the averaging condition and the averaged network satisfies

\[ l + c \gamma + hc\left(1 + l^2 \beta + c^2 \beta^2\right) < 0 \]

then, there exists a positive constant \( \varepsilon^* > 0 \) such that the consensus of agents can be reached for any \( \varepsilon \in (0, \varepsilon^*) \), where \( l > 0 \) stands for the Lipschitz constant, \( c > 0 \) is the coupling strength, \( h > 0 \) is the communication delay, and \( \beta > 0 \) is defined as \( \beta = \|G'G\| \).

- **Note:** It is a global result, and applicable for the existence of communication delay.
Numerical Simulations

- Each agent is a modified Chua’ system
- The network switches fast between the following topologies (a), (b), and (c), which leads to the averaged topology (d).

Lipschitz constant = 12
the coupling strength c = 13
delay bound h = 0.0001.

Observation:
Topology (d) can ensure leader following
Numerical Simulations

Comparison of the Consensus error

Tracking error under topology (d)

Note: theoretical upper bound $\varepsilon^* \ll 5$.

Tracking error under fast switching topology with $\varepsilon = 5$. 
Conclusions

This talk focuses on how to make a group of agents reach collective dynamics which admits nonlinear agents and delayed communications

- Leader following may be reached when the network is switching between some connective topologies even for delayed communications.
- Even when the communications between followers are directed and change over time, and some followers are isolated from time to time, they may still track the leader if some conditions can be achieved.
- When there is no leader or reference signal, if some agents cannot communicate with the others intermittently, consensus may still be achievable.
- When fast switching network occurs, if the averaged network can ensure consensus and the switches happens fast enough, consensus is also possible even for disconnected network.
Thanks

Q&A