HW7, due to 12/24

1. For $1 \le i \le n+1$, let

$$U_i^{\pm} := \left\{ (x^1, \cdots, x^{n+1}) \in \mathbb{S}^n \subset \mathbb{R}^{n+1} : \pm x^i > 0 \right\}$$

and define

$$\varphi_i^{\pm} : U_i^{\pm} \to \mathbb{R}^n, \quad (x^1, \cdots, x^{n+1}) \mapsto (x^1, \cdots, \widehat{x^i}, \cdots, x^{n+1})$$

where $\widehat{x^i}$ means that we remove x^i. Show that for $1 \le i < j \le n+1$ we have

$$\varphi_i^{\pm} \circ (\varphi_j^{\pm})^{-1}(u^1, \cdots, u^n) = \left(u^1, \cdots, \widehat{u^i}, \cdots, \pm\sqrt{1 - |\boldsymbol{u}|^2}, \cdots, u^n \right)$$

where $\boldsymbol{u} = (u^1, \cdots, u^n)$ and $\pm\sqrt{1 - |\boldsymbol{u}|^2}$ is on the j-th coordinate.

2. Show that the general linear group

$$\mathbf{GL}(n, \mathbb{R}) := \{ n \times n \text{ real matrices} \}$$

is an n^2-dimensional manifold.

3 (* Bonus). Suppose that $f \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ and define

$$M_f := \{ \boldsymbol{x} \in \mathbb{R}^{n+1} : f(\boldsymbol{x}) = 0 \}.$$

If the gradient of f

$$\text{grad}(f)(\boldsymbol{x}) := \left(\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^{n+1}} \right)(\boldsymbol{x}), \quad \boldsymbol{x} \in M_f,$$

is nonzero, then M_f is an n-dimensional smooth manifold.