HW5, due to 11/16/2020

Exercise 1. Prove (2)-(5) of Theorem 1.3.3 in Lecture Note.

Exercise 2. Prove (2) and (5) of Theorem 1.3.10 in Lecture Note.

Exercise 3 (Bonus). Let $X = (X_{ij}) \in \mathbf{Max}(n, \mathbf{R})$. Define formally

$$e^X := \sum_{m \geq 0} \frac{X^m}{m!}, \quad ||X|| := \left(\sum_{1 \leq i,j \leq n} |X_{ij}|^2 \right)^{1/2}.$$

Regarding $\mathbf{Max}(n, \mathbf{R})$ as $\mathbf{R}^{n^2}$, we say that a sequence $\{X_m\}_{m \geq 1}$ in $\mathbf{Max}(n, \mathbf{R})$ **converges** to X, if $||X_m - X|| \to 0$ as $m \to \infty$.

(a) Show that for any $X \in \mathbf{Max}(n, \mathbf{R})$, the series $\sum_{m \geq 0} X^m/m!$ converges so that the function e^X is well-defined. Moreover, show that e^X is continuous in X.

(b) Compute e^X and e^Y for

$$X = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} b & c \\ 0 & b \end{bmatrix}$$

Verify that $XY \neq YX$ and $e^X e^Y \neq e^Y e^X$.

(c) The Heisenberg group H is defined to be

$$H := \left\{ X = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in \mathbf{Max}(3, \mathbf{R}) \middle| a, b, c \in \mathbf{R} \right\} \subseteq \mathbf{SL}(3, \mathbf{R}).$$

For any $X, Y \in H$ show that $[X, [X, Y]] = [Y, [X, Y]] = 0$, where $[X, Y] := XY - YX$.

(d) Verify

$$e^X e^Y = e^{X + Y + \frac{1}{2}[X, Y]}$$

for

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Y = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$