DIFFERENTIAL MANIFOLDS: HW3 (DUE TO 2021/5/14)

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Remark: Please solve problems as many as you can!

1. Let (θ, φ) denote standard spherical coordinates on $\mathbb{S}^2 \subset \mathbb{R}^3_{(x,y,z)}$ defined by

$$\theta := \cos^{-1}(z), \quad \varphi := \tan^{-1}(y/x).$$

Then the standard metric on $\mathbb{S}^2$ can be written as

$$g_{\mathbb{S}^2} = d\theta \otimes d\theta + \sin^2\theta \, d\varphi \otimes d\varphi.$$

Fix a constant $M > 0$. Let $M := \mathbb{R}^2 \times \mathbb{S}^2$ with coordinates (t, r, θ, φ) and consider the Schwarzschild metric

$$g = -\left(1 - \frac{2M}{r}\right) dt \otimes dt + \left(1 - \frac{2M}{r}\right)^{-1} dr \otimes dr + r^2 g_{\mathbb{S}^2}$$

with $r > 2M$. Show that $\mathrm{Ric}(g) = 0$ in the region $r > 2M$.

2. Given a smooth vector field V on a Riemannian manifold (M, g). For any smooth function f we consider a diffusion operator

$$\Delta_V f := \Delta f + \langle V, \nabla f \rangle.$$

Here $\Delta = g^{ij}\nabla_i \nabla_j$ stands for the Laplace-Beltrami operator of g. For any smooth function u on M, show that

$$\frac{1}{2}\Delta_V |\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}_V(\nabla u, \nabla u) + \langle \nabla \Delta u, \nabla u \rangle,$$

where $\mathrm{Ric}_V := \mathrm{Ric} - \frac{1}{2}\mathscr{L}_V g$ denotes the Bakry-Emery-Ricci tensor field.

3. In the notation of **2**, show that if $u = u(t, x)$ is a smooth function on (M, g) satisfying $\partial_t u = \Delta_V u$, then

$$(\partial_t - \Delta_V)\frac{|\nabla u|^2}{u} = -\frac{2}{u}\left[\left|\nabla^2 u - \frac{1}{u}\nabla u \otimes \nabla u\right|^2 + \mathrm{Ric}_V(\nabla u, \nabla u)\right].$$

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