Exercise 3.3

A1

(1) $f(x)$ 的定义域为 $(0,+\infty)$. $\forall x_0 \in (0,+\infty)$, 下证 $f(x)$ 在 $x=x_0$ 处连续.

若 $|x-x_0|<\frac{1}{2}x_0$ 得到 $\frac{1}{2}x_0<x<\frac{3}{2}x_0 \Rightarrow$

$$|f(x)-f(x_0)| = \left|\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x_0}}\right| = \frac{|\sqrt{x}-\sqrt{x_0}|}{\sqrt{x}\sqrt{x_0}}$$

$$< \frac{1}{\sqrt{\frac{1}{2}x_0}}\left|\sqrt{x}-\sqrt{x_0}\right| = \frac{\sqrt{2}}{x_0}\frac{|x-x_0|}{\sqrt{x}+\sqrt{x_0}}$$

$$< \frac{\sqrt{2}}{x_0}\cdot\frac{|x-x_0|}{\sqrt{\frac{1}{2}x_0}+\sqrt{x_0}} = \frac{\sqrt{2}}{1+\sqrt{\frac{1}{2}}}\cdot\frac{1}{x_0^{3/2}}|x-x_0|$$

$$= \frac{2}{\sqrt{2}+1}\frac{1}{x_0^{3/2}}|x-x_0|$$

~~对~~ $\forall \varepsilon>0$, 取 $\delta = \min\left\{\frac{1}{2}x_0, \frac{\sqrt{2}+1}{2}x_0^{3/2}\varepsilon\right\} \Rightarrow |f(x)-f(x_0)|<\varepsilon$. 只要 $|x-x_0|<\delta$.

(2) $f(x)$ 定义域为 $(-\infty,+\infty)$.

① $\forall x_0 \in (0,+\infty)$, 下证 $f(x)$ 在 $x=x_0$ 处连续. 若 $|x-x_0|<\frac{1}{2}x_0$ 得到 $\frac{1}{2}x_0<x<\frac{3}{2}x_0 \Rightarrow$

$$|f(x)-f(x_0)| = \left|\frac{x-x_0}{x^{2/3}+x^{1/3}x_0^{1/3}+x_0^{2/3}}\right| < \frac{|x-x_0|}{\left[\left(\frac{1}{2}\right)^{2/3}+\left(\frac{1}{2}\right)^{1/3}+1\right]x_0^{2/3}}$$

故 $\forall \varepsilon>0$, 取 $\delta = \min\left\{\frac{1}{2}x_0, \left[\left(\frac{1}{2}\right)^{2/3}+\left(\frac{1}{2}\right)^{1/3}+1\right]x_0^{2/3}\varepsilon\right\}$, $\Rightarrow$ 只要 $|x-x_0|<\delta$ 有 $|f(x)-f(x_0)|<\varepsilon$.

② $x_0=0$. 此时 $|f(x)-f(0)|=|f(x)|=\sqrt[3]{|x|}$. 故 $\forall \varepsilon>0$, 取 $\delta=\varepsilon^3$

$\Rightarrow |f(x)-f(0)|<\varepsilon$, 只要 $|x-0|<\delta$.

③ $x_0\in(-\infty,0)$. 此时 $f(x)=\sqrt[3]{-x}=-\sqrt[3]{x}$, $\forall x<0$. 若 $|x-x_0|<\frac{|x_0|}{2}$

得到 $\frac{3}{2}x_0<x<\frac{1}{2}x_0 \Longrightarrow$

$$|f(x)-f(x_0)|=\frac{|x-x_0|}{|x^{2/3}+x^{1/3}x_0^{1/3}+x_0^{2/3}|}$$

$$<\frac{|x-x_0|}{\left[\left(\frac{1}{2}\right)^{2/3}+\left(\frac{1}{2}\right)^{1/3}+1\right]x_0^{2/3}}$$

$\forall\varepsilon>0$, 取 $\delta=\min\left\{\frac{1}{2}|x_0|,\left[\left(\frac{1}{2}\right)^{2/3}+\left(\frac{1}{2}\right)^{1/3}+1\right]x_0^{2/3}\varepsilon\right\} \Longrightarrow$ 只要 $|x-x_0|<\delta$, 有 $|f(x)-f(x_0)|<\varepsilon$.

A2

见讲义

A3

假设 f 与 g 两个都在 $x=0$ 处连续 $\Longrightarrow$ fg 也在 $x=0$ 处连续 $\Longrightarrow$

$$f(0)g(0)=(fg)(0)=\lim_{x\to0}(fg)(x)$$

$\forall x\neq0$, 有 $f(x)\equiv g(x) \Longrightarrow$

$$f(0)g(0)=\lim_{x\to0}[f(x)]^2=\lim_{x\to0}[g(x)]^2$$

即 $f(0)g(0)=f(0)^2=g(0)^2 \Longrightarrow f(0)=g(0)$, 矛盾!

A4

(1) $\forall x\in I$, $\exists r_n\in I\cap\mathbb{Q}$ 满足 $\lim_{n\to\infty}r_n=x$. f 连续 $\Longrightarrow$

$$f(x)=f\left(\lim_{n\to\infty}r_n\right)=\lim_{n\to\infty}f(r_n)=0$$

故 $f(x)\equiv0$, $\forall x\in I$.

(2) $\forall x_1<x_2$, $\exists r_{1,n}\in\mathbb{Q}$ 和 $r_{2,n}\in\mathbb{Q}$ 满足 $\lim_{n\to\infty} r_{1,n}=x_1$ 且 $r_{1,n}<r_{2,n}$ $\lim_{n\to\infty} r_{2,n}=x_2$. 根据条件得到

$$f(r_{1,n})<f(r_{2,n}) \quad (\text{因为 } r_{1,n}<x_1<x_2<r_{2,n})$$

从而 $f(x_1)=f(\lim_{n\to\infty} r_{1,n})=\lim_{n\to\infty} f(r_{1,n})\le \lim_{n\to\infty} f(r_{2,n})=f(x_2)$.

即 f 递增. 下证 f 必严格递增. 因为 $x_1<x_2$. $\exists r_1, r_2\in\mathbb{Q}\cap I$ 满足 $x_1<r_1<r_2<x_2 \Rightarrow f(r_1)<f(r_2)$. 但 $r_{1,n}\to x_1$, $r_{2,n}\to x_2$ 得到

$$f(r_{1,n})<f(r_1)<f(r_2)<f(r_{2,n})$$

求极限得到 $f(x_1)\le f(r_1)<f(r_2)\le f(x_2)$. □

A5

(1) $\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{\sin x}{x}=1 \Rightarrow 0$ 是可去间断点

(2) $\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{x}{\sin x}=1 \Rightarrow 0$ 是可去间断点. 但 $k\pi(k\neq 0)$ 是第二类间断点.

(3) $\lim_{x\to 0+}\frac{x(x-2)}{x(x^2-4)}=\lim_{x\to 0+}\frac{1}{x+2}=\frac{1}{2}=f_+(0)$

$f_-(0)=\lim_{x\to 0-}\frac{x(x-2)}{-x(x^2-4)}=\lim_{x\to 0-}\frac{-1}{x+2}=-\frac{1}{2}$ } 0是跳跃间断点.

$f(2)=\lim_{x\to 2}\frac{x^2-2x}{|x|(x^2-4)}=\lim_{x\to 2}\frac{x(x-2)}{x(x^2-4)}=\frac{1}{4} \Rightarrow 2$ 是可去间断点

$f(-2)=\lim_{x\to -2}\frac{x(x-2)}{-x(x^2-4)}=-\lim_{x\to -2}\frac{1}{x+2} \Rightarrow -2$ 是第2类间断点.

(4) $f(x)=\begin{cases}1 & 2k\pi<x<2k\pi+\pi\\ 0 & x=2k\pi \text{ 或 } (2k+1)\pi\\ -1, & 2k\pi+\pi<x<2k\pi+2\pi\end{cases} \Rightarrow k\pi(k\in\mathbb{Z})$ 是跳跃间断点.

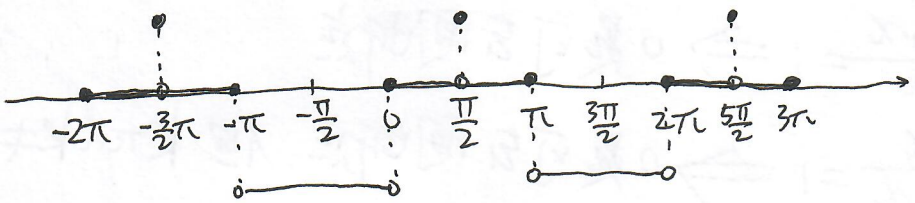
(5) $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}[x]\sin\frac{1}{x}=0 \Rightarrow 0$ 是可去间断点.

$$f_-(n)=\lim_{x\to n-}[x]\sin\frac{1}{x}=\lim_{x\to n-}(n-1)\sin\frac{1}{x}=(n-1)\sin\frac{1}{n}\ (n\neq 0)$$

$$f_+(n)=\lim_{x\to n+}[x]\sin\frac{1}{x}=\lim_{x\to n+}(n+1)\sin\frac{1}{x}=(n+1)\sin\frac{1}{n}\ (n\neq 0)$$

故 $x=n\ (n\neq 0)$ 是跳跃间断点.

(6)
$$f(x)=\begin{cases}1, & x=2k\pi+\frac{\pi}{2}\\ 0, & 2k\pi\le x<2k\pi+\frac{\pi}{2} \text{ 或 } 2k\pi+\frac{\pi}{2}<x\le 2k\pi+\pi\\ -1, & 2k\pi+\pi<x<2k\pi+2\pi\end{cases}$$



$x=2k\pi+\frac{\pi}{2}\ (k\in\mathbb{Z})$,

$x=k\pi\ (k\in\mathbb{Z})$

都是跳跃间断点.

(7) $x=0$ 是第2类间断点.

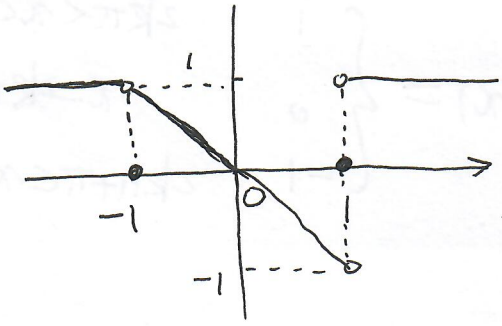
$$f_+(1)=\lim_{x\to 1+}\frac{1}{1-e^{\frac{x}{1-x}}}=\frac{1}{1-\lim\limits_{x\to 1+}e^{\frac{x}{1-x}}}=\frac{1}{1-e^{-\infty}}=1$$

$$f_-(1)=\lim_{x\to 1-}\frac{1}{1-e^{\frac{x}{1-x}}}=\frac{1}{1-\lim\limits_{x\to 1-}e^{\frac{x}{1-x}}}=\frac{1}{1-e^{+\infty}}=0$$

$\Rightarrow 1$ 是跳跃间断点.

(8) 0 是连续点，其它点都是第2类间断点.

(9)
$$f(x)=\begin{cases}1 & |x|>1\\ 0 & x=1\\ 0 & x=-1\\ -x & |x|<1\end{cases}$$



-1：可去

1：跳跃

(10) $\lim\limits_{x\to 0} f(x) = 0 = f(0)$, $\lim\limits_{x\to 1} f(x) = f(1) = 0 \Longrightarrow$ 0和1是连续点.

$\forall x_0 \neq 0, 1$:

$$\lim_{\substack{x\to x_0 \\ x\in\mathbb{Q}}} f(x) = \lim_{\substack{x\to x_0 \\ x\in\mathbb{Q}}} x(x-1) = x_0(x_0-1)$$

$$\lim_{\substack{x\to x_0 \\ x\notin\mathbb{Q}}} f(x) = \lim_{\substack{x\to x_0 \\ x\notin\mathbb{Q}}} x(x-1)\cdot 0 = 0$$

$\Longrightarrow x = x_0 (x_0 \neq 0, 1)$ 是第2类间断点.

(11) $\left.\begin{aligned} f_-(-2) &= \lim\limits_{x\to -2^-} f(x) = -\infty \\ f_+(-2) &= \lim\limits_{x\to -2^+} f(x) = -4 \end{aligned}\right\} \Longrightarrow -2$ 是第2类间断点.

$$\left.\begin{aligned} f_-(2) &= \lim_{x\to 2^-} f(x) = 0 = f(2) \\ f_+(2) &= \lim_{x\to 2^+} f(x) = \lim_{x\to 2} (x-2)\sin\frac{1}{x-2} = 0 \end{aligned}\right\}$$ 2是连续点.

(12) $f_-(0) = 0$, $f_+(0) = \lim\limits_{x\to 0^+}\left[e^{-2} - (1-2x)^{\frac{1}{x}}\right] = e^{-2} - \lim\limits_{x\to 0^+} e^{\frac{\ln(1-2x)}{x}} = e^{-2} - e^{-2} = 0$

0是可去间断点.

$$f_-\left(\frac{1}{\pi}\right) = \lim_{x\to \frac{1}{\pi}-0} f(x) = e^{-2} - \lim_{x\to \frac{1}{\pi}-0} (1-2x)^{\frac{1}{x}} = e^{-2} - \left(1-\frac{2}{\pi}\right)^{\pi} = f\left(\frac{1}{\pi}\right)$$

$$f_+\left(\frac{1}{\pi}\right) = \lim_{x\to \frac{1}{\pi}+0} \tan\frac{1}{2x} = +\infty$$

$\frac{1}{\pi}$是第2类间断点.

$$\left.\begin{aligned} f_-(1) &= \tan\frac{1}{2} = f(1) \\ f_+(1) &= \lim_{x\to 1^+} (x-1)\sin\frac{1}{x-1} = 0 \end{aligned}\right\}$$ 1是跳跃间断点

A6

(1) $\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1}\dfrac{(x-1)(x^2+x+1)}{x-1}=3$

定义 $F(x)=\begin{cases}\dfrac{x^3-1}{x-1}, & x\neq 1\\ 3, & x=1\end{cases}$

(2) $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}=\dfrac{1}{2}$

定义 $F(x)=\begin{cases}\dfrac{1-\cos x}{x^2}, & x\neq 0\\ \dfrac{1}{2}, & x=0\end{cases}$

A7

(1) 对 $\varepsilon=\dfrac{f(x_0)-g(x_0)}{2}>0$, $\exists \delta_1>0, \delta_2>0$ 满足 $|f(x)-f(x_0)|<\varepsilon$

只要 $|x-x_0|<\delta_1$, 且 $|g(x)-g(x_0)|<\varepsilon$ 只要 $|x-x_0|<\delta_2$. 见1

只要 $|x-x_0|<\delta=\min\{\delta_1,\delta_2\}$, 有

$f(x)>f(x_0)-\varepsilon \Rightarrow g(x_0)+\varepsilon>g(x)$.

(2) $f(x)>g(x) \xrightarrow[\text{限}]{\text{求极}} f(x_0)=\lim\limits_{x\to x_0} f(x)\geq \lim\limits_{x\to x_0} g(x)=g(x_0)$

A8

见讲义

A9

提示 + A8

A10

否则 $\exists x, y \in [a,b]$ 满足 $f(x)<0$ 且 $f(y)>0$. 不妨假设 $x<y$. 根据介值定理, $\exists \xi \in (x,y)$ 使得 $f(\xi)=0$ 成立, 矛盾!

A11

见讲义

A12

见讲义

A13

定义 $F(x) \triangleq f(x) - \sum_{1\le k\le n} \lambda_k f(x_k) \in C([a,b])$

$$= \sum_{1\le k\le n} \lambda_k \left[f(x) - f(x_k) \right]$$

$$\Rightarrow F(x_i) = \sum_{1\le i,k\le n} \lambda_k \left[f(x_i) - f(x_k) \right]$$

$$\Rightarrow F(x_i) = f(x_i) - \sum_{1\le k\le n} \lambda_k f(x_k)$$

如果 $\forall i$, $f(x_i) - \sum_{1\le k\le n} \lambda_k f(x_k) > 0 \Rightarrow$

$$0 < \sum_{1\le i\le n} \lambda_i f(x_i) - \sum_{1\le k\le n} \lambda_k f(x_k) = 0, \text{ 矛盾!}$$

如果 $\forall i$, $f(x_i) - \sum_{1\le k\le n} \lambda_k f(x_k) < 0 \Rightarrow$ 也得到矛盾!

故$\exists i,j \in \{1,\cdots,n\}$使得

$$f(x_i)-\sum_{1\le k\le n}\lambda_k f(x_k)\ge 0 \text{ 且 } f(x_j)-\sum_{1\le k\le n}\lambda_k f(x_k)\le 0$$

即 $F(x_i)F(x_j)\le 0$. 根据介值定理 $\exists \xi$ 介于 x_i 与 x_j 间使得 $F(\xi)=0$ 成立.

A14

$$0=\frac{a_1}{x-\lambda_1}+\frac{a_2}{x-\lambda_2}+\frac{a_3}{x-\lambda_3} \Longleftrightarrow 0=a_1(x-\lambda_2)(x-\lambda_3)+a_2(x-\lambda_1)(x-\lambda_3)+a_3(x-\lambda_1)(x-\lambda_2)$$

定义 $f(x)=a_1(x-\lambda_2)(x-\lambda_3)+a_2(x-\lambda_1)(x-\lambda_3)+a_3(x-\lambda_1)(x-\lambda_2)$

$f(\lambda_1)=a_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)>0,\ f(\lambda_2)<0,\ f(\lambda_3)>0$

介值定理 $\Rightarrow \exists \xi\in(\lambda_1,\lambda_2)$ 满足 $f(\xi)=0$,

$\exists \eta\in(\lambda_2,\lambda_3)$ 满足 $f(\eta)=0$.

A15

(1) $e^{\lim\limits_{x\to 0} 2\cos x \ln(1+e^x)} = e^{2\ln 2}$

(2) $e^{\lim\limits_{x\to 0}\frac{\ln(x+e^x)}{x}} = e^{\lim\limits_{x\to 0}\frac{\ln(1+x+e^x-1)}{x+e^x-1}\cdot\frac{x+e^x-1}{x}}$

$= e^{1\times\lim\limits_{x\to 0}(1+\frac{e^x-1}{x})} = e^{1\times(1+1)} = e^2$

(3) $e^{\lim_{x\to 0} \frac{\ln\left(\frac{1+x}{1-x}\right)}{x}} = e^{\lim_{x\to 0} \frac{\ln\left(1+\frac{2x}{1-x}\right)}{\frac{2x}{1-x}} \cdot \frac{2}{1-x}} = e^2$

(4) $e^{\lim_{x\to+\infty} \frac{\ln\left(\frac{3x+2}{3x-1}\right)}{\frac{1}{2x-1}}} = e^{\lim_{y\to 0+} \frac{\ln\left(1+\frac{6y}{3+y}\right)}{y}}$

$= e^{\lim_{y\to 0+} \frac{6}{3+y}} = e^2$

(5) $e^{\lim_{x\to a} \frac{\ln \frac{\sin x}{\sin a}}{x-a}} = e^{\lim_{x\to a} \frac{\ln\left(1+\frac{\sin x-\sin a}{\sin a}\right)}{\frac{\sin x-\sin a}{\sin a}} \cdot \frac{\sin x-\sin a}{\sin a(x-a)}}$

$= e^{\frac{\cos a}{\sin a}} = e^{\cot a}$

(6) $\tan^n\left(\frac{\pi}{4}+\frac{1}{n}\right) = \left[\frac{\sin\left(\frac{\pi}{4}+\frac{1}{n}\right)}{\cos\left(\frac{\pi}{4}+\frac{1}{n}\right)}\right]^n$

$= \left[\frac{\sin\frac{\pi}{4}\cos\frac{1}{n} + \cos\frac{\pi}{4}\sin\frac{1}{n}}{\cos\frac{\pi}{4}\cos\frac{1}{n} - \sin\frac{\pi}{4}\sin\frac{1}{n}}\right]^n = \left(\frac{\cos\frac{1}{n}+\sin\frac{1}{n}}{\cos\frac{1}{n}-\sin\frac{1}{n}}\right)^n$

$\Rightarrow e^{\lim_{n\to\infty} n \ln \frac{1+\tan\frac{1}{n}}{1-\tan\frac{1}{n}}}$

② $e^{\lim_{x\to 0} \frac{\ln\frac{1+\tan x}{1-\tan x}}{x}} = e^{\lim_{x\to 0} \frac{\ln\left(1+\frac{2\tan x}{1-\tan x}\right)}{x}}$

$= e^{\lim_{x\to 0} \frac{2\tan x}{x}\cdot\frac{1}{1-\tan x}} = e^2$

A16

$\forall x, y \geq 1 \Rightarrow$

$$|f(x)-f(y)| = |\sqrt{x}-\sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{|x-y|}{2}$$

A17

见讲义

A18

见讲义

B19

~~$f(x)=\sqrt{x}$ 在 $[0,1]$ 上连续，根据 Cantor 定理 $f(x)$ 在 $[0,1]$ 上一致连续。由 A16，$f(x)$ 在 $[1,+\infty)$ 上一致连续。$\forall \varepsilon > 0$，只要 $|x-y|<\delta$，$\exists \delta = \min\{2\varepsilon, 1\}$，$\forall x, y \geq 0$ 有~~

$f(x)$ 在 $[0,2]$ 上连续 $\Rightarrow \forall \varepsilon > 0 \exists \delta_1 > 0 \forall x, y \in [0,2]$ 只要 $|x-y| < \delta_1$ 都有 $|f(x)-f(y)| < \varepsilon$（利用 Cantor 定理）。根据 A16，$f(x)$ 在 $[1,+\infty)$ 上一致连续，对上述 $\varepsilon > 0$ $\exists \delta_2 > 0$ $\forall x, y \geq 1$ 只要 $|x-y| < \delta_2$ 都有 $|f(x)-f(y)| < \varepsilon$。现取 $\delta = \min\{\delta_1, \delta_2, 1\}$。则 $\forall x, y \geq 0$ 只要 $|x-y| < \delta$，则要么 $x, y \geq 1$ 且 $|x-y| < \delta_2$，要么 $x, y \in [0,2]$ 且 $|x-y| < \delta_1$，无论何情况都有 $|f(x)-f(y)| < \varepsilon$。

B20

$$|f(x)-f(y)| = |\cos\sqrt{x} - \cos\sqrt{y}|$$

$$= 2\left|\sin\frac{\sqrt{x}+\sqrt{y}}{2}\sin\frac{\sqrt{x}-\sqrt{y}}{2}\right| \leq |\sqrt{x}-\sqrt{y}|$$

由B19，知道 $\forall \varepsilon>0 \exists \delta>0, \forall x,y\geq 0$ 只要 $|x-y|<\delta$ 有 $|\sqrt{x}-\sqrt{y}|<\varepsilon$.
从而 $|f(x)-f(y)|<\varepsilon$.

B22

给定 $x\in(a,b)$，$\forall y<x$ 有 $f(y)\leq f(x) \Rightarrow f_+(x)$ 存在. 类似可证 $f_+(x)$ 存在. 若 $x=a$，则 $f_+(a)$ 存在；若 $x=b$，则 $f_-(b)$ 存在. 故 $\forall x\in[a,b]$，$f_+(x)$ 或 $f_-(x)$ 都存在，且若 $f_+(x)$ 和 $f_-(x)$ 都存在，则 $f_-(x)\leq f_+(x)$. 假设 f 在 $x=x_0$ 处不连续

① $x_0\in(a,b)$：此时 $f_-(x)<f_+(x)$. 若 $f_-(x)=f(x)$，则 $[f(x),f_+(x)]$ 取不到值. 若 $f_+(x)=f(x)$ 则 $[f_-(x),f(x)]$ 取不到值，矛盾！

② $x_0=a$：此时 $f(a)<f_+(a) \Rightarrow [f(a),f_+(a)]$ 取不到值.

③ $x_0=b$：此时 $f_-(b)<f(b) \Rightarrow [f_-(b),f(b)]$ 取不到值.

综上所述，f 在 $[a,b]$ 上必连续.

B23

(1) $a_{n+1}=f(a_n) \Rightarrow a_n>0$ 且 $a_n\downarrow \Rightarrow \lim\limits_{n\to\infty} a_n$ 存在

(2) 对 $a_{n+1}=f(a_n)$ 两边求极限并利用 f 是连续的 $\Rightarrow t=f(t)$

(3) 若 $t>0 \Rightarrow 0\leq f(t)<t$，矛盾. 故 $t=0$.

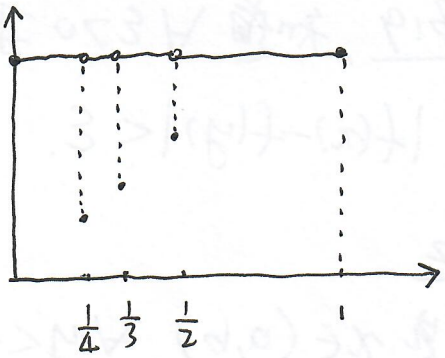
C24

(1) 是

(2) 不一定，见讲义

C25

(1) $$f(x)=\begin{cases}\frac{1}{2}, & x=\frac{1}{2}\\ \frac{1}{3}, & x=\frac{1}{3}\\ \frac{1}{4}, & x=\frac{1}{4}\\ 1 & x\in[0,1]\setminus\{\frac{1}{2},\frac{1}{3},\frac{1}{4}\}\end{cases}$$



(2) $$f(x)=\begin{cases}(x-\frac{1}{2})(x-\frac{1}{3})(x-\frac{1}{4}), & x\in\mathbb{Q}\\ -(x-\frac{1}{2})(x-\frac{1}{3})(x-\frac{1}{4}), & x\notin\mathbb{Q}\end{cases}$$

(3) $$f(x)=\begin{cases}\frac{1}{n}, & x=\frac{1}{n}\\ 2, & x\in[0,1]\setminus\{\frac{1}{n}:n=1,2,\cdots\}\end{cases}$$

(4) $$f(x)=\begin{cases}x, & x\in[0,1]\cap\mathbb{Q}\\ 2x, & x\in[0,1]\setminus\mathbb{Q}\end{cases}$$

(5) $$f(x)=\begin{cases}0, & x=0\\ \frac{1-x}{x}, & x\in(0,1]\end{cases}$$