**HW1**

**1.** Page 83, Exercise 1, 3, 8.

**2 (Bonus).** A **(commutative) ring** $A$ is a set with two binary operations $(+, \cdot)$ such that

1) $(A, +)$ is an Abelian group (so that $A$ has an 0 element, and for any $x \in A$ there is a unique (additive) inverse $-x \in A$).
2) $x \cdot (y+z) = x \cdot y + x \cdot z$, $(y+z) \cdot x = y \cdot x + z \cdot x$, and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for any $x, y, z \in A$.
3) $x \cdot y = y \cdot x$ for any $x, y \in A$.
4) There exists a unique element $1 \in A$ such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in A$.

A map $f : A \to B$ between two rings is said to be a **ring homomorphism** if it satisfies

1) $f(x+y) = f(x) + f(y)$, for all $x, y \in A$.
2) $f(x \cdot y) = f(x) \cdot f(y)$, for all $x, y \in A$.
3) $f(1) = 1$.

Assume that $A$ is a ring.

(1) An **ideal** $\mathfrak{a}$ of $A$ is a subset of $A$ which is an additive subgroup and is such that $A\mathfrak{a} \subseteq \mathfrak{a}$ (i.e., $xy \in \mathfrak{a}$ for all $x \in A$ and $y \in \mathfrak{a}$). Then $A/\mathfrak{a}$ is a ring and $\phi : A \to A/\mathfrak{a}$, $x \mapsto x + \mathfrak{a}$, is a ring homomorphism.

(2) An ideal $\mathfrak{p}$ in $A$ is **prime** if $\mathfrak{p} \neq A$ and if $(xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ or $y \in \mathfrak{p})$.

If $f : A \to B$ is a ring homomorphism and $\mathfrak{q}$ is a prime ideal in $B$, then $f^{-1}(\mathfrak{q})$ is a prime ideal in $A$.

Let

$$X := \{\text{primes ideals of } A\}, \quad V(\mathfrak{a}) := \{\mathfrak{p} \in X | \mathfrak{a} \subseteq \mathfrak{p}\}$$

where $\mathfrak{a}$ is an ideal.

(a) Prove that
- $V(0) = X$, $V(A) = \emptyset$.
- $\mathfrak{a} \subseteq \mathfrak{b} \Rightarrow V(\mathfrak{a}) \supseteq V(\mathfrak{b})$.
- $V(\sum_{\in I} \mathfrak{a}_i) = \cap_{i \in I} V(\mathfrak{a}_i)$ for any family of ideals $\mathfrak{a}_i$, $i \in I$, of $A$.
- $V(\mathfrak{a}) \cup V(\mathfrak{a}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$.

Here $\mathfrak{a}\mathfrak{b}$ is the ideal generated by all products $x \in \mathfrak{a}$ and $y \in \mathfrak{y}$, that is, $\mathfrak{a}\mathfrak{b} = \{\sum_{1 \leq i \leq n} x_i y_i : x_i \in \mathfrak{a},\ y_i \in \mathfrak{b}\}$.

We then obtain the **Zariski topology** $(X, \mathscr{T}) =:$ Spec$(A)$ on $X$, where

$$\mathscr{T} := \{U \in X : X \setminus U = V(\mathfrak{a}) \text{ for some ideal } \mathfrak{a} \text{ of } A\}.$$

(b) We have $X = \emptyset$ if and only if $0 = 1$ in $A$.

(c) For any ideal $\mathfrak{a}$ of $A$, define the **nilpotent radical** of $\mathfrak{a}$ to be the ideal

$$\sqrt{\mathfrak{a}} := \{a \in A : a^n \in \mathfrak{a} \text{ for some positive integer } n\}.$$

Then we have $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

(d) $\sqrt{\mathfrak{a}} = \cap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$.

(e) For any ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $A$, we have $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} \subseteq V(\mathfrak{b})$.