HW6, due to 12/5

1. Page 171, Exercise 2(a), 3, 5, 7.

* **Bonus**

A **direct set** is a partially ordered set $(I, \leq)$ such that for any $i, j \in I$, there is a $k \in I$ such that $i, j \leq k$. A **direct system** $(A_i, \phi_{ij})_{i \in I}$ (I is a direct set) of sets consists of a family of sets A_i and maps $\phi_{ij} : A_i \to A_j$ for pairs $i \leq j$ such that

(i) $\phi_{ii} = \mathrm{id}_{A_i}$,

(ii) $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ whenever $i \leq j \leq k$.

Let $(A_i, \phi_{ij})_{i \in I}$ be a direct system. For $x_i \in A_i$ and $x_j \in A_j$, we say that x_i is equivalent to x_j, written as $x_i \sim x_j$, if there is a $k \geq i, j$ such that

$$\phi_{ik}(x_i) = \phi_{jk}(x_j) \in A_k.$$

(1) Show that $\sim$ is an equivalence relation on $\bigsqcup_{i \in I} A_i$.

The **direct limit** is

$$\operatorname{dir.}\lim_{i \in I} A_i := \bigsqcup_{i \in I} A_i / \sim = \{[x_i] : i \in I\}.$$

Let X be a topological space and $x \in X$. For any two neighborhoods U and V of x, we say $V \leq U$ if $U \subseteq V$. Let

$$\mathcal{N}_x := \{\text{all neighborhoods of } x \text{ in } X\}.$$

(2) Show that $(\mathcal{N}_x, \leq)$ is a direct set.

For any presheaf $\mathscr{F}$ on X, define the **stalk** $\mathscr{F}_x$ **of** $\mathscr{F}$ **at** x by

$$\mathscr{F}_x := \operatorname{dir.}\lim_{x \in U} \mathscr{F}(U) = \{[(s, U)] : s \in \mathscr{F}(U),\ U \in \mathcal{N}_x\}.$$

(3) Show that two sections $s \in \mathscr{F}(U)$ and $t \in \mathscr{F}(V)$ define the same element in $\mathscr{F}_x$ if and only if there is a neighborhood W of x such that $W \subseteq U \cap V$ and $s|_W = t|_W$.

For any $U \in \mathcal{N}_x$, we have a canonical map

$$\mathscr{F}(U) \longrightarrow \mathscr{F}_x, \quad s \longmapsto [(s, U)] := s_x$$

where s_x is called the **germ of** s **at** x.

Let $\phi : \mathscr{F} \to \mathscr{G}$ denote a morphism of presheaves between presheaves on X. For any $x \in X$, ϕ induces a homomorphism on stalks:

$$\phi_x : \mathscr{F}_x \longrightarrow \mathscr{G}_x, \quad [(s, U)] \longmapsto [(\phi(U)(s), U)].$$

(4) Show that ϕ_x is well-defined for every $x \in X$.

We say a morphism $\phi : \mathscr{F} \to \mathscr{G}$ of presheaves is **isomorphic** if there exists a morphism $\theta : \mathscr{G} \to \mathscr{F}$ of presheaves such that $\phi \circ \theta = \mathrm{id}_{\mathscr{G}}$ and $\theta \circ \phi = \mathrm{id}_{\mathscr{F}}$.

(5) Show that $\phi : \mathscr{F} \to \mathscr{G}$ is an isomorphism between presheaves if and only if for every open subset $U \subseteq X$, $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is an isomorphism.

We can define morphisms of sheaves as morphisms of presheaves.

(6) Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a topological space X. Show that ϕ is an isomorphism if and only if the induced map on stalks $\phi_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism for every point $x \in X$.