# Basic analysis

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CHAPTER 1

# Introduction

## 1.1. Sets and mappings

The **power set** of a given set $A$ is defined to be

$$2^A := \{\text{all subsets of } A\} \supseteq A.$$

Notions:

- $A := B$: $A$ is defined by $B$,
- $\forall$: any/for any,
- $\exists$: there exists/exist $\cdots$,
- !: unique,
- $\exists!$: there exists a unique $\cdots$,
- $\mathbb{N} := \{0, 1, 2, 3, \cdots\}$: the set of all national numbers,
- $\mathbb{Z}$: the set of all integers,
- $\mathbb{Z}_{<0} := \mathbb{Z} \setminus \mathbb{N}$,
- $\mathbb{Q}$: the set of all rational numbers,
- $\mathbb{R}$: the set of all real numbers,
- $\mathbb{C}$: the set of all complex numbers,
- i.e.: (Latin) id est = that is/in other words,
- e.g.: (Latin) exempli gratia = for example,
- WLOG: without loss of generality,
- TFAE: the following are equivalent,
- resp.: respectively,

**1.1.1. Arbitrary unions and intersections.** Let $\mathscr{A}$ be a collection of sets.

(1) **Union**:

$$\bigcup_{A \in \mathscr{A}} A := \{x : x \in A \text{ for at least one } A \in \mathscr{A}\}.$$

(2) **Intersection**:

$$\bigcap_{A \in \mathscr{A}} A := \{x : x \in A \text{ for every } A \in \mathscr{A}\}.$$

When $\mathscr{A} = \emptyset$, we let $\cup_{A \in \mathscr{A}} A = \emptyset$.

**1.1.2. Cartesian products I.** Let $A, B$ be two sets.

(1) **Cartesian product**:

$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}.$$

(2) **Order pair**:

$$(a, b) := \{\{a\}, \{a, b\}\}$$

where $a$ is called the **first coordinate** while $b$ the **second coordinate**.

**1.1.3. Maps.** Let $C, D$ be two sets.

(1) A **rule of assignment** is a subset $R$ of $C \times D$ such that

$$(c, d) \in R \text{ and } (c, d') \in R \implies d = d'.$$

(2) Suppose that $R$ is a rule of assignment. Define

$$\begin{aligned} \mathbf{Dom}(R) \equiv \mathbf{domain}(R) &:= \{c \in C : \exists\, d \in D \text{ such that } (c, d) \in R\}, \\ \mathbf{Im}(R) \equiv \mathbf{image}(R) &:= \{d \in D : \exists\, c \in C \text{ such that } (c, d) \in R\}. \end{aligned}$$

A **map** $f$ is a pair $(R, B)$, where $R$ is a rule of assignment and $B$ is a set (called the **range** of $f$), such that $\mathbf{Im}(R) \subset B$.

(1) **domain** of $f \equiv \mathbf{Dom}(f) := \mathbf{Dom}(R)$,
(2) **image** of $f \equiv \mathbf{Im}(f) := \mathbf{Im}(R)$,
(3) We write:

$$f : A \longrightarrow B, \quad a \longmapsto f(a),$$

where $A$ is the domain of $f$, $B$ is the range of $f$ (so that $\mathbf{Im}(f) \subseteq B$), and $f(a)$ is the unique element of $B$ satisfying $(a, f(a)) \in R$.

**Example 1.1.1.** Assume $C = D := \mathbb{R}$, $f(x) := x^2$, $R := \mathbb{R} \times \mathbb{R}_{\geq 0}$, and $B := R$. In this case, $A = \mathbb{R}$ and $\mathbf{Im}(f) = \mathbb{R}_{\geq 0}$.

Consider two maps $f : A \to B$ and $g : B \to C$.

(1) For a given subset $A_0$ of $A$, define the **restriction of** $f$ **to** $A_0$ as the map $f|_{A_0} = f : A_0 \to B$.
(2) **Composite**:

$$g \circ f : A \longrightarrow C, \quad a \longmapsto c$$

where $f(a) = b$ and $g(b) = c$ for some $b \in B$.

Suppose that $f : A \to B$ is a map.

(1) $f$ is **injective** if

$$f(a) = f(a') \implies a = a'.$$

(2) $f$ is **surjective** if

$$\forall\, b \in B\ \exists\, a \in A \text{ such that } f(a) = b.$$

(3) $f$ is **bijective** if $f$ is injective and surjective.
(4) If $f$ is bijective, we define its **inverse** $f^{-1}$ by

$$f^{-1}(b) = a \iff f(a) = b.$$

**Lemma 1.1.2.** *Let $f : A \to B$ be a map. If there exist a* **left inverse** *$g : B \to A$ of $f$ (i.e., $g(f(a)) = a$ for all $a \in A$) and a* **right inverse** *$h : B \to A$ of $f$ (i.e., $f(h(b)) = b$ for all $b \in B$), then $f$ is bijective and $g = h = f^{-1}$.*

**Exercise 1.1.3.** (1) Show that if $f$ has a left (resp., right) inverse, then $f$ is injective (resp., surjective).

(2) Given examples of maps that have a left (resp., right) inverse but no right (resp., left) inverse.

(3) Can a map have more than one left (or right) inverse?

(4) Prove Lemma 1.1.2.

Let $f : A \to B$ be a map, $A_0 \subseteq A$ and $B_0 \subseteq B$.

(1) **Image of** $A_0$ **under** $f \equiv f(A_0) := \{f(a) : a \in A_0\}$,
(2) **Preimage of** $B_0$ **under** $f \equiv f^{-1}(B_0) := \{a : f(a) \in B_0\}$.
(3) It is clear that

$$A_0 \subseteq f^{-1}(f(A_0)), \quad B_0 \supseteq f(f^{-1}(B_0)).$$

There are examples such that both equalities in (3) may not be true (Find such examples!).

When $B$ is a number field, we will say **functions** instead of maps.

**1.1.4. Categories.** A **category** $\mathfrak{C}$ consists of

1) a family $\mathbf{Ob}(\mathfrak{C})$ of **objects** of $\mathfrak{C}$,
2) $\forall$ pair $(X, Y)$ of $\mathbf{Ob}(\mathfrak{C})$, $\exists$ set $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$ of **morphisms** from $X$ to $Y$, and
3) $\forall$ triple $(X, Y, Z)$ of $\mathbf{Ob}(\mathfrak{C})$, $\exists$ map

$$\mathrm{Hom}_{\mathfrak{C}}(X, Y) \times \mathrm{Hom}_{\mathfrak{C}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathfrak{C}}(X, Z), \quad (f, g) \longmapsto g \circ f$$

called the **composition map**.

These data satisfy

a) composition is associative, i.e., $(f \circ g) \circ h = f \circ (g \circ h)$,
b) $\forall X \in \mathbf{Ob}(\mathfrak{C})\ \exists\ \mathrm{id}_X \in \mathrm{Hom}_{\mathfrak{C}}(X, X)$ such that

$$f \circ \mathrm{id}_X = f, \quad \mathrm{id}_X \circ g = g$$

for any $f \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathfrak{C}}(Y, X)$.

**Example 1.1.4.** There are some classical categories:

(1) **Set**: sets and functions,
(2) **Group**: groups and group homomorphisms (in abstract algebra),
(3) $\mathbf{Vect}_{\mathbb{R}}$: real vector spaces and $\mathbb{R}$-linear maps (in linear algebra),
(4) **Top**: topological spaces and continuous maps (in topology),
(5) **Calabi-Yau category**: in differential geometry/algebraic geometry $\rightsquigarrow$ homological mirror symmetry/SYZ conjecture.

Let $\mathfrak{C}$ be a category.

(1) Write $f \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ as $f : X \to Y$.
(2) $f \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ is an **isomorphism** if there exists a morphism $g \in \mathrm{Hom}_{\mathfrak{C}}(Y, X)$ such that

$$f \circ g = \mathrm{id}_Y \quad \text{and} \quad g \circ f = \mathrm{id}_X.$$

(3) A **subcategory** $\mathfrak{C}'$ of $\mathfrak{C}$ is a category such that
   - $\mathbf{Ob}(\mathfrak{C}') \subseteq \mathbf{Ob}(\mathfrak{C})$,
   - $\mathrm{Hom}_{\mathfrak{C}'}(X, Y) \subseteq \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ for all $X, Y \in \mathbf{Ob}(\mathfrak{C}')$,
   - $\mathrm{id}_X \in \mathrm{Hom}_{\mathfrak{C}'}(X, X)$ for each $X \in \mathbf{Ob}(\mathfrak{C}')$.

   We say that $\mathfrak{C}'$ is a **full subcategory** of $\mathfrak{C}$ if it is a subcategory and moreover $\mathrm{Hom}_{\mathfrak{C}'}(X, Y) = \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ for each pair $(X, Y)$ of objects.

(4) **Top** is a subcategory, but not a full subcategory, of **Set**.

(5) The **opposite category** $\mathfrak{C}^\circ$ of $\mathfrak{C}$ is defined as follows:

$$\mathbf{Ob}(\mathfrak{C}^\circ) := \mathbf{Ob}(\mathfrak{C}), \quad \mathrm{Hom}_{\mathfrak{C}^\circ}(X, Y) := \mathrm{Hom}_{\mathfrak{C}}(Y, X).$$

(6) Let $f \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$.
   - $f$ is a **monomorphism** or is said to be **injective** if for any $W \in \mathbf{Ob}(\mathfrak{C})$ and any $g, g' \in \mathrm{Hom}_{\mathfrak{C}}(W, X)$ with $f \circ g = f \circ g'$, we have $g = g'$.

$$W \xrightarrow[g']{g} X \xrightarrow{f} Y$$

   - $f$ is an **epimorphism** or is said to be **surjective** if for any $Z \in \mathbf{Ob}(\mathfrak{C})$ and any $h, h' \in \mathrm{Hom}_{\mathfrak{C}}(Y, Z)$ with $h \circ f = h' \circ f$, we have $h = h'$.

$$X \xrightarrow{f} Y \xrightarrow[h']{h} Z$$

   - $f$ is said to be **bijective** if it is both injective and surjective.

(7) $P \in \mathbf{Ob}(\mathfrak{C})$ is **initial** if for any $Y \in \mathbf{Ob}(\mathfrak{C})$, $\mathrm{Hom}_{\mathfrak{C}}(P, Y)$ has exactly one element. $Q \in \mathbf{Ob}(\mathfrak{C})$ is **final** if for any $X \in \mathbf{Ob}(\mathfrak{C})$, $\mathrm{Hom}_{\mathfrak{C}}(X, Q)$ has exactly one element.

**Exercise 1.1.5.** (1) Prove that two initial (resp., final) objects are isomorphic.
(2) Isomorphism is bijective, but the converse may not true.

A **(covariant) functor** $F : \mathfrak{C} \to \mathfrak{C}'$ between two categories consists
1) a map $F : \mathbf{Ob}(\mathfrak{C}) \to \mathbf{Ob}(\mathfrak{C}')$,
2) $\forall$ pair $(X, Y)$ in $\mathbf{Ob}(\mathfrak{C})$, $\exists$ map $F : \mathrm{Hom}_{\mathfrak{C}}(X, Y) \to \mathrm{Hom}_{\mathfrak{C}'}(F(X), F(X'))$.

These data satisfy
a) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and
b) $F(f \circ g) = F(f) \circ F(g)$.

A **contravariant functor** $G : \mathfrak{C} \to \mathfrak{C}'$ is a functor $G : \mathfrak{C}^\circ \to \mathfrak{C}'$.

**Example 1.1.6.** (1) **Forgetful factor** $F$ : **Top** $\to$ **Set**.
(2) **Fundamental group functor** $\pi_1$ : $\mathbf{Top}_*$ $\to$ **Group**, $(X, x) \mapsto \pi_1(X, x)$ (the fundamental group of $X$ at $x$).
(3) For a given $X \in \mathbf{Ob}(\mathfrak{C})$, define

$$\mathrm{Hom}_{\mathfrak{C}}(X, \cdot) : \mathfrak{C} \longrightarrow \mathbf{Set}, \quad Z \longmapsto \mathrm{Hom}_{\mathfrak{C}}(X, Z),$$
$$\mathrm{Hom}_{\mathfrak{C}}(\cdot, X) : \mathfrak{C} \longrightarrow \mathbf{Set}, \quad Z \longmapsto \mathrm{Hom}_{\mathfrak{C}}(Z, X).$$

Then

$$\mathrm{Hom}_{\mathfrak{C}}(X, \cdot) \text{ is covariant and } \mathrm{Hom}_{\mathfrak{C}}(\cdot, X) \text{ is contravariant.}$$

Consider two functors $F_1, F_2 : \mathfrak{C} \to \mathfrak{C}'$. A **morphism** or **natural transformation** $\theta : F_1 \to F_2$ consists of

$$X \in \mathbf{Ob}(\mathfrak{C}) \Longrightarrow \theta(X) \in \mathrm{Hom}_{\mathfrak{C}'}(F_1(X), F_2(X)).$$

These data satisfy the following diagram

$$\begin{array}{ccc} F_1(X) & \xrightarrow{\theta(X)} & F_2(X) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\ F_1(Y) & \xrightarrow[\theta(Y)]{} & F_2(Y) \end{array} \qquad F_2(f) \circ \theta(X) = \theta(Y) \circ F_1(f),$$

is commutative, i.e., $F_2(f) \circ \theta(X) = \theta(Y) \circ F_1(f)$, for any $X, Y \in \mathbf{Ob}(\mathfrak{C})$ and $f \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$.

**Definition 1.1.7.** Given two categories $\mathfrak{C}$ and $\mathfrak{C}'$, define a new category $\mathbf{Func}(\mathfrak{C}, \mathfrak{C}')$ with

$$\mathbf{Ob}\left(\mathbf{Func}(\mathfrak{C}, \mathfrak{C}')\right) := \{\text{functors } F : \mathfrak{C} \to \mathfrak{C}'\}$$

and

$$\mathrm{Hom}_{\mathbf{Func}(\mathfrak{C}, \mathfrak{C}')}(F_1, F_2) := \{\text{morphisms } \theta : F_1 \to F_2\}.$$

**Definition 1.1.8.** Let $\mathfrak{C}$ be a category. We say that $F : \mathfrak{C} \to \mathbf{Set}$ is a **representable functor** if $\exists\, X \in \mathbf{Ob}(\mathfrak{C})$ such that $F$ is isomorphic to $\mathrm{Hom}_{\mathfrak{C}}(X, \cdot)$ in the category $\mathbf{Func}(\mathfrak{C}, \mathbf{Set})$.

**Remark 1.1.9.** If $F : \mathfrak{C} \to \mathbf{Set}$ is representable, then $X$ is unique up to isomorphism and is called a **representative of** $F$.

**Definition 1.1.10.** We say a functor $F : \mathfrak{C} \to \mathfrak{C}'$ is **fully faithful** if $\forall\, X, Y \in \mathbf{Ob}(\mathfrak{C})$, the map $\mathrm{Hom}_{\mathfrak{C}}(X, Y) \to \mathrm{Hom}_{\mathfrak{C}'}(F(X), F(Y))$ is bijective.

**Theorem 1.1.11.** **(Yoneda's lemma)** *(1) For any* $X \in \mathbf{Ob}(\mathfrak{C})$ *and* $F \in \mathbf{Ob}(\mathfrak{C}^{\vee})$, *where* $\mathfrak{C}^{\vee} := \mathbf{Func}(\mathfrak{C}^{\circ}, \mathbf{Set})$, *we have*

$$\mathrm{Hom}_{\mathfrak{C}^{\vee}}(\mathrm{Hom}_{\mathfrak{C}}(X), F) \simeq F(X)$$

*in* $\mathbf{Set}$, *where* $\mathrm{Hom}_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}^{\vee}$ *is a functor given by* $\mathrm{Hom}_{\mathfrak{C}}(X) := \mathrm{Hom}_{\mathfrak{C}}(\cdot, X)$.

*(2)* $\mathrm{Hom}_{\mathfrak{C}}$ *is a fully faithful functor.*

PROOF. (1) To $f \in \mathrm{Hom}_{\mathfrak{C}^\vee}(\mathrm{Hom}_{\mathfrak{C}}(X), F)$, we associate $\phi(f) \in F(X)$ as follows:

$$f(X) : \mathrm{Hom}_{\mathfrak{C}}(X, X) \longrightarrow F(X), \quad \mathrm{id}_X \longmapsto \phi(f) := f(X)(\mathrm{id}_X).$$

Conversely, to $s \in F(X)$, we can associate $\psi(s) \in \mathrm{Hom}_{\mathfrak{C}^\vee}(\mathrm{Hom}_{\mathfrak{C}}(X), F)$ as follows:

$$\mathrm{Hom}_{\mathfrak{C}}(Y, X) \xrightarrow{\ F\ } \mathrm{Hom}_{\mathbf{Set}}(F(X), F(Y)) \xrightarrow{\ s\ } F(Y)$$

with $\psi(s)(Y) := s \circ F$. Then $\phi$ and $\psi$ are inverses to each other.

(2) For any $X, Y \in \mathbf{Ob}(\mathfrak{C})$, one has

$$\mathrm{Hom}_{\mathfrak{C}^\vee}(\mathrm{Hom}_{\mathfrak{C}}(X), \mathrm{Hom}_{\mathfrak{C}}(Y)) \simeq \mathrm{Hom}_{\mathfrak{C}}(\cdot, Y)(X) = \mathrm{Hom}_{\mathfrak{C}}(X, Y)$$

which implies that $\mathrm{Hom}_{\mathfrak{C}}$ is fully faithful. □

**1.1.5. Relations.** A **relation** on a set $A$ is a subset $C$ of $A \times A$. If $C$ is a relation on $A$, then we write $xCy$ to be $(x, y) \in C$.

An **equivalence relation** on a set $A$ is a relation $C$ on $A$ having the following properties:

a) (**Reflexivity**) $\forall\, x \in A \Longrightarrow xCx$,
b) (**Symmetry**) $xCy \Longrightarrow yCx$,
c) (**Transitivity**) $xCy$ and $yCz \Longrightarrow xCz$.

Notion: $\sim :=$ equivalence relation.

(1) The **equivalence class** of $x \in A$:
$$[x] := \{y \in A : y \sim x\} \ni x.$$
(2) Two equivalence classes are either disjoint or equal. Hence
$$A = \bigcup \{[x] : x \in A\}.$$
(3) A **partition** of a set $A$ is a collection of disjoint nonempty subsets of $A$ whose union is $A$.
(4) Given a partition $\mathscr{D}$ of $A$, there exists an equivalence relation $\sim$ on $A$ from which it is derived.

Indeed, define $\sim$ on $A$ by requiring $x \sim y$ if and only if $x, y$ belong to the same element of $\mathscr{D}$. Then $\sim$ is an equivalence relation on $A$. Assume that $\sim$ and $\sim'$ are two equivalence relations on $A$ that give rise to the same collection of equivalence classes $\mathscr{D}$. Given $x \in A$, let
$$[x] := \{y \in A : y \sim x\}, \quad [x]' := \{y \in A : y \sim' x\}.$$
Because $[x] \cap [x]' \ni \{x\}$, we must have $[x] = [x]'$.

A relation $C$ on a set $A$ is called an **order relation** or **simple order** or **linear order**, if it has the following properties:

a) (**Comparability**) $\forall\, x, y \in A$ with $x \neq y \Longrightarrow$ either $xCy$ or $yCx$,
b) (**Nonreflexivity**) there does not exist $x \in A$ such that $xCx$,
c) (**Transitivity**) $xCy$ and $yCz \Longrightarrow xCz$.

Notion: $< :=$ order relation.

(1) Equivalently:
 a) $x \neq y \Longrightarrow$ either $x < y$ or $y < x$,
 b) $x < y \Longrightarrow x \neq y$,
 c) $x < y$ and $y < z \Longrightarrow x < z$.

(2) $x \leq y$ means $x < y$ or $x = y$.
(3) Let $(X, <)$ be an order relation. For $a < b$, define

$$(a, b) := \{x \in X : a < x < b\}$$

called an **open interval** in $X$. If $(a, b) =$, then $a$ is the **immediate predecessor** of $b$, and $b$ is the**immediate successor** of $a$.
(4) Consider two sets with order relations $(A, <_A)$ and $(B, <_B)$. We say $A$ and $B$ have the same **order type** if there exists a bijective correspondence between them that preserves orders. That is, there exists a bijective function $f : A \to B$ such that

$$a_1 <_A a_2 \Longrightarrow f(a_1) <_B f(a_2).$$

For example, $((-1, 1), <)$ and $(\mathbb{R}, <)$ have the sane order type ($x \mapsto \frac{x}{1-x^2}$); $(\{0\} \cup (1, 2), <)$ and $([0, 2), <)$ have the same order type ($0 \mapsto 0$ and $x \mapsto x - 1$ for $1 < x < 2$).
(5) Let $(A, <_A)$ and $(B, <_B)$ be two sets with order relations. Define an order relation $<$ on $A \times B$ by

$$(a_1, b_1) < (a_2, b_2)$$

if $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$.

Assume that $(A, <)$ is a set with order relation, and $A_0$ is a subset of $A$.

(1) $b$ is the **largest number** of $A_0$ if $b \in A_0$ and if $x \leq b$ for any $x \in A_0$. $a$ is the **smallest number** of $A_0$ if $a \in A_0$ and if $a \leq x$ for any $x \in A_0$.
(2) $A_0$ is **bounded above** if there is a $b \in A$ such that $x \leq b$ for all $x \in A_0$. We call $b$ is an **upper bound** for $A_0$. Let

$$\sup(A_0) := \begin{matrix}\text{the smallest element among} \\ \text{all upper bounds for } A_0,\end{matrix}$$

be the **least upper bound** or **supremum**.

$A_0$ is **bounded below** if there is a $a \in A$ such that $a \leq x$ for all $x \in A_0$. We call $a$ is a **lower bound** for $A_0$. Let

$$\inf(A_0) := \begin{matrix}\text{the largest element among} \\ \text{all lower bounds for } A_0,\end{matrix}$$

be the **greatest lower bound** or **infimum**.
(3) $(A, <)$ is said to have the **least upper bound property** (or shortly **LUBP**) if any nonempty subset $A_0$ of $A$ that is bounded above has a least upper bound. Similarly, it is said to have the **greatest lower bound property** (or shortly **GLBP**) if any nonempty subset $A_0$ of $A$ that is bounded below has a greatest lower bound. Observe that **LUBP** $\Leftrightarrow$ **GLBP**.

The set $B := (-1, 0) \cup (0, 1)$ does not have the least upper bound property (check it!).

Given a set $A$, a relation $\prec$ on $A$ is called a **strict partial order** on $A$ if it has the following properties:

1) (**Non-reflexivity**) $a \prec a$ never hold,
2) (**Transitivity**) $a \prec b$ and $b \prec c \Rightarrow a \prec c$.

On $\mathbb{R}^2$, there is a natural strict partial order $\prec$ defined by

$$(x_0, y_0) \prec (x_1, y_1) \iff y_0 = y_1 \text{ and } x_0 < x_1.$$

Let $A$ be a set with a strict partial order $\prec$.

(1) If $B \subseteq A$, an **upper bound** on $B$ is an element $c$ of $A$ such that for any $b \in B$, either $b = c$ of $b \prec c$.

(2) A **maximal element** of $A$ is an element $m$ of $A$ such that for no element $a$ of $A$ does the relation $m \prec a$ hold.

(3) **Zorn's lemma** (1935):

> Let $A$ be a set that is strictly partially ordered. If any simply ordered subset of $A$ has an upper bound in $A$, then $A$ has a maximal element.

One of applications of Zorn's lemma is as follows: Let $A = \{a_n\}_{n \geq 1}$ with $a_i \in \mathbb{R}$ and $|a_i| \leq M$ for some positive number $M$. Then $(A, <)$ is a set with the strict partial order $<$. By Zorn's lemma, $A$ has a maximal element.

In general, consider a sequence of functions $f(x,t)$ such that $|f(x,t)| \leq M$ for all $x \in [0,1]$ and $t \in \mathbb{R}$. For each $x \in [0,1]$, define

$$A_x := \{f(x,t)\}_{t \in \mathbb{R}}.$$

Then $A_x$ has a maximal element $f(x)$. Then we get a map

$$f : [0,1] \longrightarrow A := \cup_{x \in [0,1]} A_x, \quad x \longmapsto f(x).$$

What's behavior of this function $f(x)$?

**1.1.6. Cartesian products II.** Let $\mathscr{A}$ be a nonempty collection of sets. An **indexing function** for $\mathscr{A}$ is a surjective function $f$ from some set $J$, called the **index set**, to $\mathscr{A}$.

(1) We say $(\mathscr{A}, f)$ an **indexed family of sets**.

(2) Given $\alpha \in J$, denote the set $f(\alpha) \in \mathscr{A}$ by $A_\alpha$, and the indexed family of sets by $\{A_\alpha\}_{\alpha \in J}$.

(3) Define

$$\begin{aligned} \bigcup_{\alpha \in J} A_\alpha &:= \{x : \exists\, \alpha \in J \text{ such that } x \in A_\alpha\}, \\ \bigcap_{\alpha \in J} A_\alpha &:= \{x : \forall\, \alpha \in J,\ x \in A_\alpha\}. \end{aligned}$$

(4) When $J = \{1, \cdots, n\}$, we denote (3) to be

$$\bigcup_{\alpha \in J} A_\alpha = \bigcup_{1 \leq i \leq n} A_i, \quad \bigcap_{\alpha \in J} A_\alpha = \bigcap_{1 \leq i \leq n} A_i.$$

(5) When $J = \mathbb{Z}_{\geq 1}$, we denote (3) to be

$$\bigcup_{\alpha \in J} A_\alpha = \bigcap_{i \geq 1} A_i, \quad \bigcap_{\alpha \in J} A_\alpha = \bigcap_{i \geq 1} A_i.$$

Let $m \in \mathbb{N}$. Given a set $X$, we define an $m$**-tuple** of elements of $X$ to be a function

$$\boldsymbol{x} : \{1, \cdots, m\} \longrightarrow X$$

For each $i\{1, \cdots, m\}$, write

$$\boldsymbol{x}(i) := x_i,$$

$i$**-th coordinate of** $\boldsymbol{x}$, and $\boldsymbol{x} = (x_1, \cdots, x_m)$.

(1) Let $\{A_1, \cdots, A_m\}$ be a family of sets indexed with the set $\{1, \cdots, m\}$. Let

$$X := A_1 \cup \cdots \cup A_m.$$

We define the **Cartesian product** of this indexed family, denoted by

$$\prod_{1 \le i \le m} A_i,$$

to be the set of all $m$-tuples $(x_1, \cdots, x_m)$ of elements of $X$ such that $x_i \in A_i$, $1 \le i \le m$.

**Remark 1.1.12.** (1) Recall two definitions of $A \times B$:

$$\begin{aligned} A \times_1 B &:= \{(a,b) : a \in A \text{ and } b \in B\}, \\ A \times_2 B &:= \{\boldsymbol{x} : \{1,2\} \to A \cup B \text{ such that } \boldsymbol{x}(1) \in A \text{ and } \boldsymbol{x}(2) \in B\}. \end{aligned}$$

Define

$$f : A \times_1 B \longrightarrow A \times_2 B, \quad (a,b) \longmapsto f((a,b))$$

with $f((a,b))(1) = a$ and $f((a,b))(2) = b$. Science $f$ is bijective, $A \times_1 B \cong A \times_2 B$.

(2) For $A, B, C$, we have three Cartesian products

$$A \times (B \times C), \quad (A \times B) \times C, \quad A \times B \times C$$

that are bijective. In particular, we can define $A^m$ for $m \ge 1$.

Given a set $X$, define $\omega$**-tuple** of elements of $X$ to be the function

$$\boldsymbol{x} : \mathbb{Z}_{\ge 1} \longrightarrow X, \quad n \longmapsto x_n := \boldsymbol{x}(n),$$

and write $\boldsymbol{x} = (x_n)_{n \ge 1}$. Let $\{A_i\}_{i \in \mathbb{Z}_{\ge 1}}$ be a family of sets indexed with the positive integers. Let

$$X := \bigcup_{i \in \mathbb{Z}_{\ge 1}} A_i.$$

The **Cartesian product** of $\{A_i\}_{i \in \mathbb{Z}_{\ge 1}}$, denoted by

$$\prod_{i \in \mathbb{Z}_{\ge 1}} A_i,$$

is defined to be the set of $\omega$-tuples $(x_i)_{i \in \mathbb{Z}_{\ge 1}}$ of $X$ such that $x_i \in A_i$.

In general, let $J$ be an index set and $X$ a set.

(1) An $J$**-tuple** of $X$ is a function

$$\boldsymbol{x} : J \longrightarrow X, \quad \alpha \longmapsto x_\alpha := \boldsymbol{x}(\alpha),$$

where $x_\alpha$ is said to be the $\alpha$**-coordinate of** $\boldsymbol{x}$. Write $\boldsymbol{x} = (x_\alpha)_{\alpha \in J}$.

(2) Let

$$X^J := \{J\text{-tuples of elements of } X\}.$$

(3) Let $\{A_\alpha\}_{\alpha \in J}$ be an index family of sets, and $X := \cup_{\alpha \in J} A_\alpha$. The **Cartesian product** of $\{A_\alpha\}_{\alpha \in J}$, denoted by

$$\prod_{\alpha \in J} A_\alpha,$$

is defined to be the set of all $J$-tuples $(x_\alpha)_{\alpha \in J}$ of $X$ such that $x_\alpha \in A_\alpha$.

When $X_n$ are all $\mathbb{R}$, we obtain

$$\mathbb{R}^\omega := \prod_{n\geq 1} X_n.$$

**1.1.7. Finite, countable and uncountable sets.** In this subsection we study the set

$$\{0,1\}^\omega := \prod_{n\geq 1} X_n$$

where $X_n := \{0,1\}$.

**Definition 1.1.13.** A set $A$ is **finite**, if it is empty or there is a bijective correspondence of $A$ with some $\{1, \cdots, n\}$. When $A = \emptyset$ we say that $A$ has **cardinality** 0, otherwise we say that $A$ has **cardinality** $n$.

**Lemma 1.1.14.** *Suppose that $n \in \mathbb{Z}_{\geq 1}$, $A$ is a nonempty set, and $a_0 \in A$. There is a bijective correspondence $f$ of $A$ with $\{1, \cdots, n+1\}$ if and only if there is a bijective correspondence $g$ of $A \setminus \{a_0\}$ with $\{1, \cdots, n\}$.*

PROOF. $\Leftarrow$: Define $f : A \to \{1, \cdots, n+1\}$ to be

$$f(a_0) := n+1, \quad f(x) := g(x)\ (x \neq a_0).$$

$\Rightarrow$: If $f(a_0) = n+1$, then we define $g := f|_{A\setminus\{a_0\}}$. Suppose now that $f(a_0) = m \in \{1, \cdots, n\}$, and let $a_1 \in A$ be such that $f(a_1) = n+1$. Then $a_1 \neq a_0$. Define $h : A \setminus \{a_0\} \to \{1, \cdots, n\}$ to be $h(a_1) = m$ and $h(x) := f(x)$ for $x \neq a_1$. □

**Theorem 1.1.15.** *Let $A$ be a set and assume there is a bijection $f : A \to \{1, \cdots, n\}$ for some $n \in \mathbb{Z}_{\geq 1}$. If $B$ is a proper subset of $A$, then there is no bijection $g : B \to \{1, \cdots, n\}$, but (provided $B \neq \emptyset$) there is a bijection $h : B \to \{1, \cdots, m\}$ for some $m < n$.*

PROOF. WLOG, we may assume that $B \neq \emptyset$. We prove it by induction. When $n = 1$, $A = \{a\}$, and $B = \emptyset$. Suppose that the theorem is true for $n$. Let $f : A \to \{1, \cdots, n+1\}$ be a bijection and $B$ a nonempty proper subset of $A$. Take $a_0 \in B$ and $a_1 \in A \setminus B$. By Lemma 1.1.14, there is a bijection $g : A \setminus \{a_0\} \to \{1, \cdots, n\}$. Since $B \setminus \{a_0\}$ is a proper subset of $A \setminus \{a_0\}$, the inductive hypothesis implies that there is no bijection $h : B \setminus \{a_0\} \to \{1, \cdots, n\}$, and either $B \setminus \{a_0\} = \emptyset$ or there is a bijection $k : B \setminus \{a_0\} \to \{1, \cdots, m\}$ (for some $m < n$). Applying again Lemma 1.1.14, the theorem holds for $n+1$. □

**Corollary 1.1.16.** *(1) If $A$ is finite, then there is no bijection of $A$ with a proper subset of itself.*

*(2) $\mathbb{Z}_{\geq 1}$ is not finite.*

*(3) The cardinality of a finite set $A$ is uniquely determined by $A$.*

*(4) Any subset of a finite set is finite. If $B$ is a proper subset of a given finite set $A$, then the cardinality of $B$ is strictly less than the cardinality of $A$.*

*(5) $B \neq \emptyset \Rightarrow$ TFAF:*

(i) *$B$ is finite,*
(ii) *there is a surjective function from some $\{1, \cdots, n\}$ onto $B$,*
(iii) *there is an injective function from $B$ into some $\{1, \cdots, n\}$.*

*(6) Finite union and finite Cartesian products of finite sets are finite.*

PROOF. (1) Assume that $B$ is a proper subset of $A$ and there is a bijection $f : A \to B$. Because $A$ is finite, there is a bijection $g : A \to \{1, \cdots, n\}$. Then $g \circ f^{-1} : B \to \{1, \cdots, n\}$ is a bijection, which is impossible!

(2) Define the map $f : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1} \setminus \{1\}$ by $f(n) := n + 1$. Since $\mathbb{Z}_{\geq 1} \setminus \{1\}$ is proper and $f$ is bijective, it follows from (1) that $\mathbb{Z}_{\geq 1}$ can not be finite.

(3) Suppose that $f : A \to \{1, \cdots, n\}$ and $g : A \to \{1, \cdots, m\}$ are bijective, for some $m, n \in \mathbb{Z}_{\geq 1}$. Then $g \circ f^{-1} : \{1, \cdots, n\} \to \{1, \cdots, m\}$ is bijective, so that $m = n$.

(4) Clearly.

(5) $(i) \Rightarrow (ii)$ : clearly. $(ii) \Rightarrow (iii)$ : Suppose that $f : \{1, \cdots, n\} \to B$ is surjective. Define $g : B \to \{1, \cdots, n\}$ to be

$$g(b) := \text{the smallest element of } f^{-1}(\{b\}).$$

For $b \neq b'$, $f^{-1}(\{b\}) \cap f^{-1}(\{b'\}) = \emptyset$, so that $g$ is injective. $(iii) \Rightarrow (i)$ : Suppose that $g : B \to \{1, \cdots, n\}$ is injective. Then there is some $m \leq n$ such that $g : B \to \{1, \cdots, m\}$ is bijective. Then $B$ is finite.

(6) If $A$ and $B$ are finite and both are not empty. There are bijections $f : \{1, \cdots, m\} \to A$ and $g : \{1, \cdots, n\} \to B$ for some choice of $m$ and $n$. Define

$$h : \{1, \cdots, m+n\} \longrightarrow A \cup B, \quad i \longmapsto \begin{cases} f(i), & 1 \leq i \leq m, \\ g(i-m), & m+1 \leq i \leq m+n. \end{cases}$$

Since $h$ is surjective, according to (5) we see that $A \cup B$ is finite. By induction, we can prove that finite unions of finite sets is finite.

From the relation

$$A \times B := \bigcup_{a \in A} \{a\} \times B$$

which is the finite union of finite sets, we conclude that $A \times B$ and therefore finite Cartesian products of finite sets are finite. □

Unfortunately, the situation of infinite Cartesian products of finite sets is more complicated. We need the following definitions.

**Definition 1.1.17.** (1) A set $A$ is said to be **infinite** if it is not finite. It is said to be **countably infinite** if there is a bijective correspondence $f : A \to \mathbb{Z}_{\geq 1}$.

(2) A set is said to be **countable** if it is either finite or countably infinite. A set that is not countable is said to be **uncountable**.

**Theorem 1.1.18.** *$B \neq \emptyset \implies$ TFAE:*

(a) *$B$ is countable,*
(b) *there is surjective function $f : \mathbb{Z}_{\geq 1} \to B$,*
(c) *there is injective function $g : B \to \mathbb{Z}_{\geq 1}$.*

PROOF. $(a) \Rightarrow (b)$ : obvious.

$(b) \Rightarrow (c)$ : Let $f : \mathbb{Z}_{\geq 1} \to B$ be surjective. Define $g : B \to \mathbb{Z}_{\geq 1}$ by $g(b) :=$ the smallest element of $f^{-1}(\{b\})$.

$(c) \Rightarrow (a)$ : Let $g : B \to \mathbb{Z}_{\geq 1}$ be an injective function. Then there is a bijection of $B$ with subset of $\mathbb{Z}_{\geq 1}$. Hence we suffice to prove that every subset of $\mathbb{Z}_{\geq 1}$ is countable (see Lemma 1.1.19). $\square$

**Lemma 1.1.19.** *If $C$ is an infinite subset of $\mathbb{Z}_{\geq 1}$, then $C$ is countable infinite.*

PROOF. Define $h : \mathbb{Z}_{\geq 1} \to C$ a bijection as follows. Denote by $h(1)$ the smallest element of $C$. Then assuming that $h(1), \cdots, h(n-1)$ are defined. Let

$$h(n) := \text{the smallest element of } C \setminus \bigcup_{1 \leq i \leq n-1} h(i).$$

**Claim 1: $h$ is injective**. If $m < n$, then $h(m) \in h(\{1, \cdots, n-1\})$ so that $h(m) \neq h(n)$.

**Claim 2: $h$ is surjective**. Let $c \in C$. The injectivity of $h$ implies $h(\mathbb{Z}_{\geq 1})$ is infinite and therefore $h(n) > c$ for some $c \in \mathbb{Z}_{\geq 1}$. Let

$$m := \text{the smallest element of } \mathbb{Z}_{\geq 1} \text{ such that } h(m) \geq c.$$

For each $i = 1, \cdots, m-1$, we have $h(i) < c$ so that $c \in C \setminus \cup_{1 \leq i \leq m-1} h(i)$. From the definition of $h(m)$, we must have $h(m) \leq c$. Hence $h(m) = c$. $\square$

**Corollary 1.1.20.** *(1) A subset of a countable set is countable.*
*(2) $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ is countably infinite.*

PROOF. (1) Let $A \subseteq B$ and $B$ be countable. By Theorem 1.1.18, there is an injection $f : B \to \mathbb{Z}_{\geq 1}$. Then $f|_A : A \to \mathbb{Z}_{\geq 1}$ is also injective, so that $A$ is countable.

(2) Since $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ is infinite, we now construct an injection $f : \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$. Define

$$f(n, m) := 2^n 3^m.$$

If $f(n, m) = f(p, q)$, then $2^n 3^m = 2^p 3^q$. If $n < p$, then $3^m = 2^{p-n} 3^q$, contradicting! Therefore $n = p$ and then $m = q$. $\square$

**Theorem 1.1.21.** *(1) A countable union of countable sets is countable.*

*(2) A finite Cartesian product of countable sets is countable.*

*(3)* $\{0,1\}^\omega$ *is uncountable.*

*(4) Given a set* $A$*. Then there are no injection* $f: 2^A \to A$ *and surjection* $g: A \to 2^A$.

*(5)* $2^{\mathbb{Z}_{\geq 1}}$ *is uncountable.*

PROOF. Observe that (5) follows from (4) and Theorem 1.1.18.

(1) Let $\{A_n\}_{n\in J}$ be an indexed family of countable sets, where the index set $J$ is either $\{1,\cdots,N\}$ or $\mathbb{Z}_{\geq 1}$. Assume each $A_n \neq \emptyset$. By Theorem 1.1.18, there are surjections $f_n : \mathbb{Z}_{\geq 1} \to A_n$ and $g: \mathbb{Z}_{\geq 1} \to J$. Define

$$h: \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \longrightarrow \bigcup_{n\in J} A_n, \quad (k,m) \longmapsto f_{g(k)}(m).$$

Then $h$ is a surjective function.

(2 )WLOG, we may assume that the Cartesian product of two countable sets $A$ and $B$ is countable. As in (1), there are surjections $f: \mathbb{Z}_{\geq 1} \to A$ and $g: \mathbb{Z}_{\geq 1} \to B$. Define $h: \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \to A \times B$ to be $h(m,n) := (f(m), g(n))$.

(3) Let $X = \{0,1\}$. For any given function $g: \mathbb{Z}_{\geq 1} \to X^\omega$, we claim that $g$ is not surjective. Denote

$$g(n) := (x_{n1}, x_{n2}, x_{n3}, \cdots, x_{nn}, \cdots), \quad x_{ij} \in \{0,1\}.$$

Define $\boldsymbol{y} := (y_i)_{i\in\mathbb{Z}_{\geq 1}}$ by

$$y_n := \begin{cases} 0, & x_{nn} = 1, \\ 1 & x_{nn} = 0. \end{cases}$$

Then $\boldsymbol{y} \in X^\omega$ but $\boldsymbol{y} \notin g(\mathbb{Z}_{\geq 1})$.

(4) It suffices to prove that given a map $g: A \to 2^A$, the map $g$ is not surjective (because the existence of an injection implies the existence of a surjection). Define

$$B := \{a \in A : a \in A \setminus g(a)\} \in 2^A.$$

Assume that $g(a_0) = B$. Then

$$a_0 \in B \iff a_0 \in A \setminus g(a_0) \iff a_0 \in A \setminus B.$$

Hence $g$ is not surjective. □

**Exercise 1.1.22.** (1) A real number $x$ is said to be **algebraic** if it satisfies some polynomial equation of positive degree

$$0 = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{Q}.$$

Assuming that each polynomial equation has only finitely many roots, show that the set of algebraic numbers is countable.

(2) A real number is said to be **transcendental** if it is not algebraic. Assuming that $\mathbb{R}$ is uncountable, show that the transcendental numbers are uncountable (e.g., $e$, $\pi$ are transcendental).

**Exercise 1.1.23.** We say that two sets $A$ and $B$ **have the same cardinality** if there exists bijection of $A$ and $B$.

(1) Show that if $B \subseteq A$ and if there is bijection $f: A \to B$, then $A$ and $B$ have the same cardinality. [Hint: define $A_1 := A$, $B_1 := B$, and for any $n \geq 2$, $A_n := f(A_{n-1})$, $B_n := f(B_{n-1})$. Then $A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq A_3 \supseteq \cdots$. Define

$$h: A \longrightarrow B, \quad x \longmapsto \begin{cases} f(x), & \text{if } x \in A_n \setminus B_n \text{ for some } n, \\ x, & \text{otherwise.} \end{cases}$$

]

(2) (**Schroeder-Berstein theorem**) If there exist injections $A \to B$ and $B \to A$, then $A$ and $B$ have the same cardinality.

## 1.2. One variable functions

We have learned **elementary functions** in high schools:

(1) **Constant functions**: $y = c$,
(2) **Power functions**: $y = x^a$, $a \neq 0$,
(3) **Exponential functions**: $y = a^x$, $a > 0$, $a \neq 1$, $x \in \mathbb{R}$,
(4) **Logarithmic functions**: $y = \log_a x$, $a > 0$, $a \neq 1$, $x > 0$,
(5) **Trigonometric functions**: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$,
(6) **Inverse trigonometric functions**: $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $\csc^{-1} x$.

### 1.2.1. Some special type functions.

We also know, for example, periodic functions, bounded functions, even/odd functions, monotone functions, inverse functions, $\cdots$.

**Example 1.2.1.** (a) **Dirichlet function**

$$D(x) := \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

(b) **sign function**

$$\operatorname{sgn}(x) := \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

(c) Define

$$\lfloor x \rfloor := n \quad \text{if } n \leq x < n+1,$$

and

$$\langle x \rangle := x - \lfloor x \rfloor.$$

(d) Define

$$\pi(x) := \#\,(\text{prime numbers } \leq x).$$

(e) **Möbius function**

$$\mu(n) := \begin{cases} (-1)^r, & n = p_1 \cdots p_r \text{ with } p_1, \cdots, p_r \text{ distinct}, \\ 0, & \text{otherwise.} \end{cases}$$

Figure: Johann Peter Gustav Lejeune Dirichlet (1805/2/13 - 1859/5/5)

(f) **Margoldt function**

$$\Lambda(n) := \begin{cases} \ln p, & n = p^\alpha,\ \alpha \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(g) **Hyperbolic functions**:

$$\sinh x := \frac{e^x - e^{-x}}{2}, \quad \cosh x := \frac{e^x + e^{-x}}{2}, \quad \tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Observe that $y = \sin x$ (resp. $y = \sinh x$) is a solution of ODE $y'' + y = 0$ (resp. $y'' - y = 0$).

**1.2.2. Prime numbers and prime number theorem.** Let $p_1 < p_2 < \cdots$ be the sequence of all prime numbers.

**Theorem 1.2.2. (Euclid's theorem)** *There are infinitely many prime numbers.*

PROOF. Otherwise, there are finite many prime numbers $p_1 < \cdots < p_N$. Consider

$$a := p_1 \cdots p_N + 1.$$

Then there is some $p_i$ that divides $a$; consequently, $p_i | 1$, which is a contradiction. □

Basic Questions:

(A) **Are there formulas giving the $n$th prime number?** The answer is yes, and there are many! But they are useless! For example,

$$p_n = 1 + \sum_{1 \leq m \leq 2^n} \left\lfloor \left\lfloor \frac{n}{1 + \pi(m)} \right\rfloor^{1/n} \right\rfloor.$$

When $n = 2$, this formula yields

$$p_2 = 1 + \sum_{1 \le m \le 4} \left\lfloor \left\lfloor \frac{2}{1+\pi(m)} \right\rfloor^{1/2} \right\rfloor = 1+1+1+0+0 = 3.$$

(B) **Behavior or distribution of primes**. The answer is the **Prime number theorem**.

From Theorem 1.2.2, we see that

$$p_{k+1} \le p_1 + \cdots p_k + 1, \quad k \ge 1.$$

Because $p_1 = 2$, we have

$$p_k \le 2^{2^{k-1}}, \quad k \ge 1. \tag{1.2.1}$$

Indeed, by the induction hypothesis, one has

$$p_{k+1} \le \prod_{1 \le i \le k} p_i + 1 \le \prod_{1 \le i \le k} 2^{2^{i-1}} + 1 = 2^{2^k} - 1 + 1 = 2^{2^k}.$$

**Corollary 1.2.3.** *For any $x \ge 2$, we get*

$$\pi(x) > \ln \ln x. \tag{1.2.2}$$

PROOF. There is an integer $\ell \in \mathbb{Z}_{\ge 1}$ such that $2^{2^{\ell-1}} \le x < 2^{2^\ell}$. Hence $\pi(x) \ge \ell$ because $p_\ell \le 2^{2^{\ell-1}} \le x$. From $2^\ell > \ln x / \ ln 2$ we can conclude that

$$\pi(x) > \ell > \frac{\ln(\ln x / \ln 2)}{\ln 2} > \frac{\ln \ln x}{\ln 2} > \ln \ln x$$

since $0 < \ln 2 < 1$. □

By the Taylor series (we shall learn later), $(1-z)^{-1} = \sum_{n \ge 0} z^n$ ($|z| < 1$), we see

$$2 \ge \frac{p}{p-1} = \left(1 - \frac{1}{p}\right)^{-1} = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots$$

and

$$\begin{aligned} 2^{\pi(x)} &\ge \prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \le x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) \\ &\ge \sum_{n \le x} \frac{1}{n} \ge \int_1^{\lfloor x \rfloor + 1} \frac{dt}{t} = \ln(\lfloor x \rfloor + 1) > \ln x. \end{aligned}$$

Here the definite integral will be given later.

**Theorem 1.2.4.** *For any $x \ge 2$, we have*

$$\pi(x) \ge \frac{1}{2 \ln 2} \ln \lfloor x \rfloor \tag{1.2.3}$$

*and then*

$$p_n \le 4^n, \quad n \ge 1. \tag{1.2.4}$$


Figure: Leonhard Euler (1707/4/15 - 1783/9/18)

PROOF. Let $2 = p_1 < p_2 < \cdots < p_j \le x$ be all primes $\le x$. Write any $n \le x$ as

$$n = \ell^2 \cdot m$$

where $\ell$ is a positive integer and $m$ is square-free (i.e., $m = p_1^{\epsilon_1} \cdots p_j^{\epsilon_j}$, $\epsilon_1, \cdots, \epsilon_j \in \{0, 1\}$). Then $\ell \le \sqrt{x}$. There are at most $\sqrt{x}$ possibilities for $\ell$ at most $2^j$ possibilities for $m$. Hence

$$x \le \sqrt{x} 2^j \quad \Longrightarrow \quad j \ge \frac{\ln \sqrt{x}}{\ln 2} = \frac{\ln x}{2 \ln 2}.$$

Thus $\pi(x) = \pi(\lfloor x \rfloor) \ge \ln\lfloor x \rfloor / 2 \ln 2$. □

Chebyshev estimates:

(1) Leonhard Euler (1762) and Carl Friedrich Gauss (1792) conjectured:

$$\pi(x) \sim \frac{x}{\ln x}. \tag{1.2.5}$$

(2) Adrien-Marie Legendre (1798) conjectured:

$$\pi(x) \sim \frac{x}{A \ln x + B} \tag{1.2.6}$$

with (1808) $A = 1$ and $B = -1.08366$.

(3) Charles-Jean de la Vallée Poussin and Jacques Hadamard (1896) proved the prime number theorem:

$$\pi(x) \sim \frac{x}{\ln x}. \tag{1.2.7}$$

(4) **Logarithmic integral** (Gauss):

$$\mathbf{li}(x) := \text{p.v.} \int_0^x \frac{dt}{\ln t} = \lim_{\epsilon \to 0} \left( \int_0^{1-\epsilon} + \int_{1+\epsilon}^x \right) \frac{dt}{\ln t}, \quad x \ge 2, \tag{1.2.8}$$

is a good approximation for $\pi(x)$. Define

$$\mathbf{Li}(x) := \int_2^x \frac{dt}{\ln t} \quad \text{(definite integral)}. \tag{1.2.9}$$


Figure: Johann Carl Friedrich Gauss (1777/4/30 - 1855/2/23)

Figure: Adrien-Marie Legendre (1752/9/18 - 1833/1/10)

Then

$$\mathbf{Li}(x) = \underbrace{\lim_{\epsilon \to 0} \left( \int_0^{1-\epsilon} + \int_{1+\epsilon}^{2} \right) \frac{dt}{\ln t}}_{\text{well-defined improper integral}} + \mathbf{Li}(x). \tag{1.2.10}$$

Figure: Jacques Solomon Hadamard (1865/12/8 - 1963/10/17)

Figure: Pafnuty Chebyshev (1821/5/26 - 1894/12/8)


Figure: Paul Erdös (1913/3/26 - 1996/9/20)

Because

$$\lim_{\epsilon\to 0}\left(\int_0^{1-\epsilon}+\int_{1+\epsilon}^2\right)\frac{dt}{\ln t} = \lim_{\epsilon\to 0}\left(\int_0^{1-\epsilon}\frac{dt}{\ln t}+\int_0^{1-\epsilon}\frac{ds}{\ln(2-s)}\right)$$

$$= \lim_{\epsilon\to 0}\left[\int_\epsilon^1\frac{du}{\ln(1-u)}+\int_\epsilon^1\frac{du}{\ln(1+u)}\right] = \int_0^1\left[\frac{1}{\ln(1-u)}+\frac{1}{\ln(1+u)}\right]du$$

and

$$\lim_{u\to 0} u^{1/2}\left[\frac{1}{\ln(1-u)}+\frac{1}{\ln(1+u)}\right] = \lim_{u\to 0}\frac{u^{-1/2}}{2}[-(1-u)+(1+u)] = 0.$$

(5) Prime number theorem implies

$$\mathbf{li}(x)\sim\mathbf{Li}(x)\sim\frac{x}{\ln x}\sim\pi(x).$$

Since

$$\mathbf{Li}(x) = \frac{x}{\ln x}+\frac{x}{\ln^2 x}-\frac{2}{\ln 2}-\frac{2}{\ln^2 2}-2\int_2^x\frac{dt}{\ln^3 t} = \frac{x}{lnx}\left[1+\frac{1}{\ln x}+O\left(\frac{1}{\ln^2 x}\right)\right],$$

we get

$$\pi(x)\sim\frac{x}{\ln x}\frac{1}{1-\frac{1}{\ln x}} = \frac{x}{\ln x-1}.$$

(6) **Chebyshev's estimate** (1850):

– for any $x \geq 2$,

$$c_1\frac{x}{\ln x}\leq\pi(x)\leq C_1\frac{x}{\ln x}, \tag{1.2.11}$$

where $c_1 := \ln(2^{1/2}3^{1/3}5^{1/5}/30^{1/30}) \approx 0.921292$ and $C_1 = 6c_1/5 \approx 1.1055$.

– if $\pi(x)/(x/\ln x)$ has the limit as $x\to\infty$, then this limit must be 1.

**Theorem 1.2.5.** **(Erdös)** *For any $x \geq 2$, we have*

$$\frac{3\ln 2}{8}\frac{x}{\ln x}\leq\pi(x)\leq 6\ln 2\frac{x}{\ln x}. \tag{1.2.12}$$

PROOF. **Step 1:** Let $e_p(n!)$ be the exponent of which $p$ appears in the factorization of $n!$. Then

$$e_p(n!) = \sum_{k\geq 1}\left\lfloor\frac{n}{p^k}\right\rfloor$$

For example,

$$e_2(4) = e_2(2^3\times 3) = 3 = 2+1 = \lfloor 4/2\rfloor+\lfloor 4/2^2\rfloor.$$

Assume it holds for $n$ and write $n+1 = p^u m$ where $p\nmid m$. Then

$$e_p((n+1)!) = e_p(n!)+u = \sum_{1\leq k\leq u}\underbrace{\left(\left\lfloor\frac{n}{p^k}\right\rfloor+1\right)}_{\lfloor(n+1)/p^k\rfloor}+\sum_{k>u}\left\lfloor\frac{n}{p^k}\right\rfloor.$$

**Step 2:** For any $n \geq 2$, one has

$$\prod_{n<p\leq 2n} p \,\Big|\, \binom{2n}{n} \quad \text{and} \quad \binom{2n}{n} \,\Big|\, \prod_{p<2n} p^{r_p}$$

where $r_p$ is the unique integer satisfying $p^{r_p} \leq 2n < p^{r_p+1}$. Indeed, the first divisibility relation is obvious. For $p < 2n$,

$$e_p\left(\binom{2n}{n}\right) = e_p((2n)!) - 2e_p(n!) = \sum_{k\geq 1}\left(\left\lfloor \frac{2n}{p^k}\right\rfloor - 2\left\lfloor \frac{n}{p^k}\right\rfloor\right).$$

If $k > r_p$, then $p^k \geq p^{r_p+1} > 2n$ so that $\lfloor 2n/p^k \rfloor = 0 = \lfloor n/p^k \rfloor$. Therefore

$$e_p\left(\binom{2n}{n}\right) = \sum_{1\leq k\leq r_p}\left(\left\lfloor \frac{2n}{p^k}\right\rfloor - 2\left\lfloor \frac{n}{p^k}\right\rfloor\right) = \sum_{1\leq k\leq r_p} 1 = r_p$$

since $\lfloor 2y \rfloor - 2\lfloor y \rfloor = 1$ if $\frac{m}{2} \leq y \leq \frac{m+1}{2}$ for some $m \geq 0$.

**Step 3:** For any $x \geq 2$,

$$\pi(x) \geq \frac{3\ln 2}{8}\frac{x}{\ln x}.$$

By **Step 2**, $\binom{2n}{n} \leq (2n)^{\pi(2n)}$. Because

$$(1+1)^{2n} = \sum_{0\leq k\leq 2n}\binom{2n}{k} \quad \text{and} \quad \binom{2n}{n} \geq \binom{2n}{k} \; (0 \leq k \leq 2n),$$

we get

$$\binom{2n}{n} > \frac{2^{2n}}{2n+1} > 2^n, \quad n \geq 3.$$

Consequently,

$$2^n < \frac{2^{2n}}{2n+1} < \binom{2n}{n} \leq (2n)^{\pi(2n)} \quad \Longrightarrow \quad \pi(2n) > \frac{\ln 2}{2}\frac{2n}{\ln(2n)} \quad (n \geq 3).$$

Assume that $x \geq 8$ and let $n$ be the unique integer satisfying $2n \leq x < 2n+2$ (so $n \geq 3$). Moreover, $2n > x - 2 \geq \frac{3}{4}x$. Since the function $y \mapsto y/\ln y$ is increasing for any $y \geq e$ (i.e., $(y/\ln y)' = (\ln y - 1)/(\ln y)^2 \geq 0$ for $y \geq e$), we conclude that

$$\pi(x) \geq \pi(2n) \geq \frac{\ln 2}{2}\frac{2n}{\ln(2n)} \geq \frac{\ln 2}{2}\frac{3x/4}{\ln(3x/4)} = \frac{3\ln 2}{8}\frac{x}{\ln x + \ln\frac{3}{4}} > \frac{3\ln 2}{8}\frac{x}{\ln x}$$

for $x \geq 8$.

**Step 4:** For $x \geq 2$, one has

$$\pi(x) \leq 6\ln 2\frac{x}{\ln x}.$$

By **Step 2**, $\pi_{n<p\leq 2n} p < (1+1)^{2n} = 2^{2n}$ and

$$2n\ln 2 > \sum_{n<p\leq 2n}\ln p \geq \ln n[\pi(2n) - \pi(n)] = \pi(2n)\ln n - \pi(n)\left(\ln\frac{n}{2} + \ln 2\right).$$

Using $\pi(n) \leq n$ yields

$$\pi(2n)\ln n - \pi(n)\ln\frac{n}{2} < 2n\ln 2 + \pi(n)\ln 2 < (3\ln 2)n.$$

Write
$$f(x) := \pi(2n)\ln 2;$$
then
$$f(n) - f(n/2) < (3\ln 2)n.$$
Take $n = 2^i$ $(2 \le i \le k)$ and obtain
$$f(2^i) - f(2^{i-1}) < (3\ln 2)2^i.$$
Hence
$$\pi(2^{k+1})\ln(2^k) < 3\ln 2 \sum_{2\le i\le k} 2^i + \pi(4)\ln 2 < 3\ln 2 \sum_{1\le i\le k} 2^i < (3\ln 2)2^{k+1}$$
so that
$$\pi(2^{k+1}) < (6\ln 2)\frac{2^k}{\ln(2^k)}.$$
Given $x \ge 2$, choose $k \ge 1$ in such a way that $2^k \le x < 2^{k+1}$. If $x \ge 4$, then $k \ge 2$ and $2^k \ge 4 > e$. Thus $2^k/\ln(2^k) \le x/\ln x$ when $x \ge 4$. Therefore
$$\pi(x) \le \pi(2^{k+1}) < 6\ln 2\frac{2^k}{\ln(2^k)} < (6\ln 2)\frac{x}{\ln x}.$$
**Step 3** and **Step 4** give the desired result. □

Bertrand's postulate:

(1) In 1845, Joseph Bertrand proved that for any $n \le 6 \cdot 10^6$, there is a prime number in $[n, 2n]$.
(2) Bertrand conjectured that (1) was true for any $n \in \mathbb{Z}_{\ge 1}$.
(3) In 1850, Chebyshev proved (2)

**Theorem 1.2.6.** *For each $n \in \mathbb{N}$, there exists a prime number $p$ such that $n < p \le 2n$.*

PROOF. The following proof is due to Erdös.

**Step 1:** For each $n \in \mathbb{N}$,
$$\prod_{p\le n} p < 4^n.$$
WLOG, we may assume that $n \ge 3$ and the result holds for each $k-1,\cdots,n-1$. If $n$ is even, then
$$\prod_{p\le n} p = \prod_{p\le n-1} p.$$
Hence one can assume that $n$ is odd. Write $n = 2m+1$ and observe
$$\prod_{m+1<p\le 2m+1} p \,\Big|\, \binom{2m+1}{m+1}, \quad \binom{2m+1}{m+1} \le \frac{2^{2m+1}}{2} = 4^m.$$
So
$$\prod_{p\le 2m+1} p = \left(\prod_{p\le m+1} p\right)\left(\prod_{m+1<p\le 2m+1} p\right) \le 4^{m+1}\cdot 4^m = 4^{2m+1}.$$

**Step 2:** If $n \geq 3$, $p$ is prime, and $\frac{2}{3}n < p \leq n$, then

$$p \nmid \binom{2n}{n}.$$

Indeed, $p > \frac{2}{3}n \geq 2$. Because $3p > 2n$, we see that $p$ and $2p$ are the only multiplies of $p$ which are $\leq 2n$. Therefore $p^2 \parallel (2n)!$. Since

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2},$$

we conclude that $\binom{2n}{n}$ is not a multiple of $p$.

**Step 3:** Assume that $n \geq 4$, and the result is false for some $n$ (so that there are no primes in the interval $[n, 2n]$). In this step, we shall show that $n < 512$. By **Step 2**, each prime number $p$ which divides $\binom{2n}{n}$ is $\leq \frac{2}{3}n$. Let $p^\alpha \parallel \binom{2n}{n}$. Then

$$\alpha \leq r_p \quad \text{and} \quad p^{r_p} \leq 2n < p^{r_p+1}.$$

If $\alpha \geq 2$, then $p^2 \leq p^\alpha \leq 2n$ and $p \leq \sqrt{2n}$, so that

$$\begin{aligned}\binom{2n}{n} &= \prod_{p|\binom{2n}{n}} p = \left(\prod_{p\|\binom{2n}{n},\ \alpha=1} p^\alpha\right)\left(\prod_{p\|\binom{2n}{n},\ \alpha\geq 2} p^\alpha\right)\\ &\leq \prod_{p\leq 2n/3} p \cdot \prod_{p\leq\sqrt{2n}} p^{r_p} \leq 4^{2n/3}\cdot(2n)^{\sqrt{2n}}.\end{aligned}$$

Using $\binom{2n}{n} \geq 2^{2n}/(2n+1)$ yields

$$4^{2n/3}(2n)^{\sqrt{2n}} \geq \frac{4^n}{2n+1} \Rightarrow 4^{n/3} \leq (2n)^{\sqrt{2n}+2} \Rightarrow \frac{\ln 2}{3}(2n) < (\sqrt{2n}+2)\ln(2n).$$

Setting $y := \sqrt{2n}$, we get

$$\frac{\ln 2}{3}y^2 - 2(y+2)\ln y < 0.$$

Consider the function $f(y) := \frac{\ln 2}{3}y^2 - 2(y+2)\ln y$ with $y \geq 0$. From

$$f'(y) = \frac{2\ln 2}{3}y - 2\ln y - 2\frac{y+2}{y}, \quad f''(y) = \frac{2}{3}\ln 2 - \frac{2}{y} + \frac{4}{y^2} > \frac{2}{3}\ln 2 - \frac{2}{y},$$

we see that when $y > 32$, $f''(y) > 0$. Since $f'(32) = \frac{64}{3}\ln 2 - 2\ln(32) - 2.2 > 0$, we obtain that $f'(y) > 0$ for $y \geq 32$. In particular, $f(y) \geq f(32)$ for $y \geq 32$. But

$$f(32) = 2^{10}\frac{\ln 2}{3} - 340 \times \ln 2 = \frac{1024-1020}{3}\ln 2 = \frac{4}{3}\ln 2 > 0,$$

we conclude that $f(y) > 0$ for any $y \geq 32$. This contradiction shows $y < 32$ or $n < 512$.

For each $n = 1, \cdots, 511$, the interval $[n, 2n]$ always contains a prime number. Therefore in **Step 4** the assumption is wrong. Thus the result holds. $\square$

Twin prime conjecture:

(1) Theorem 1.2.6 implies

$$p_{n+1} - p_n \leq p_n. \tag{1.2.13}$$


Figure: Yitang Zhang (1955 - )

**Conjecture 1.2.7. (Gramer, 1936)** *One has*

$$\limsup_{n\to\infty} \frac{p_{n+1}-p_n}{(\ln p_n)^2} \leq 1. \tag{1.2.14}$$

*The sup/inf limit will be defined later.*

(2) Baker-Haman-Pintz (2001) proved

$$p_{n+1} - p_n < p_n^{0.525}, \quad n \gg 1. \tag{1.2.15}$$

(4) If $p$ and $p+2$ are both prime numbers, we say $(p, p+2)$ is **twin prime**.

**Conjecture 1.2.8. (Twin prime conjecture)** *There exist infinitely many integers n such that $p_{n+1} - p_n = 2$. Equivalently*

$$\liminf_{n\to\infty} (p_{n+1} - p_n) = 2. \tag{1.2.16}$$

(4) Goldston-Pintz-Yildrim (2009-2010) proved

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\ln p_n} = 0, \quad \liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\sqrt{\ln p_n}(\ln\ln p_n)^2} < \infty. \tag{1.2.17}$$

**Theorem 1.2.9. (Y.-T. Zhang, 2013)** *One has*

$$\liminf_{n\to\infty} (p_{n+1} - p_n) \leq 7 \times 10^7. \tag{1.2.18}$$

Let $b_1 < \cdots < b_k$ be positive integers. For each prime number $p$, set

$$\nu_{b_1,\cdots,b_l}(p) := \#\{b_i \pmod p : 1 \leq i \leq k\}. \tag{1.2.19}$$

Figure: Godfrey Harold Hardy (1877/2/7 - 1947/12/1)

When $k = 2$, $(b_1, b_2) = (0, 2)$, we have

$$\nu_{0,2}(p) = \#\{0 \pmod p,\ 2 \pmod p\} = \begin{cases} 1, & p = 2, \\ 2, & p \geq 3. \end{cases} \quad \Longrightarrow \nu_{0,2}(p) < p.$$

**Conjecture 1.2.10. (Dickson, 1904)** *If $\nu_{b_1,\cdots,b_k}(p) < p$ for all prime numbers $p$, then there exist infinitely many positive integers $n$ such that $n + b_1, \cdots, n + b_k$ are all prime numbers.*

It is clear that Conjecture 1.2.10 implies Conjecture 1.2.8.

**Conjecture 1.2.11. (Hardy-Littlewood, 1923)** *For any $x, y \geq 1$, we have*

$$\pi(x + y) \leq \pi(x) + \pi(y). \tag{1.2.20}$$

Hensley-Richards (1972) proved that Conjecture 1.2.10 and Conjecture 1.2.11 are incompatible. People believe that Conjecture 1.2.10 is true, while Conjecture 1.2.11 would be false.

**1.2.3. $\pi$ and $e$.** As we will prove later that

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \sum_{k\geq 0}\frac{1}{k!}, \tag{1.2.21}$$

$$\frac{\pi^2}{6} = \sum_{n\geq 1}\frac{1}{n^2}, \tag{1.2.22}$$

$$n! \sim n^n e^{-n}\sqrt{2\pi n}, \quad n \to \infty. \quad (Stirling's formula) \tag{1.2.23}$$

**1.2.4. Metric spaces.** Consider the $n$ dimensional Euclidean space $\mathbb{R}^n$ with the usual distance function $d_{\mathbb{R}^n}$ given by

$$d_{\mathbb{R}^n}(\boldsymbol{x},\boldsymbol{y}) := \left(\sum_{1\le i\le n}(x^i-y^i)^2\right)^{1/2}, \quad \boldsymbol{x}=(x^1,\cdots,x^n), \quad \boldsymbol{y}=(y^1,\cdots,y^n).$$

In high school it is well known that

- $d_{\mathbb{R}^n}(\boldsymbol{x},\boldsymbol{y})\ge 0$ and $d_{\mathbb{R}^n}(\boldsymbol{x},\boldsymbol{y})=0$ if and only if $\boldsymbol{x}=\boldsymbol{y}$,
- $d_{\mathbb{R}^n}(\boldsymbol{x},\boldsymbol{y})=d_{\mathbb{R}^n}(\boldsymbol{y},\boldsymbol{x})$,
- $d_{\mathbb{R}^n}(\boldsymbol{x},\boldsymbol{z})\le d_{\mathbb{R}^n}(\boldsymbol{x},\boldsymbol{y})+d_{\mathbb{R}^n}(\boldsymbol{y},\boldsymbol{z})$.

**Definition 1.2.12.** A **metric space** is a pair $(X,d)$, where $X$ is nonempty and $d$ is a **metric** on $X$. That is, $d: X\times X\to\overline{\mathbb{R}}:=\overline{\mathbb{R}}\cup\{\infty\}$ satisfying

1) **(Positiveness)** $d(x,y)\ge 0$ and $d(x,y)=d(y,x)$,
2) **(Symmetry)** $d(x,y)=d(y,x)$,
3) **(Triangle inequality)** $d(x,z)\le d(x,y)+d(y,z)$.

We say that $d$ is **finite** if the image of $d$ is contained in $\mathbb{R}$.

Any metric space induces a finite metric on some set. Indeed, let $(X,d)$ be a metric space, and pick a point $x\in X$. Define

$$[x]_d := \{y\in X: d(x,y)\ne\infty\}.$$

Then $y\sim_d x\Leftrightarrow y\in[x]_d$ is an equivalence relation. Then $d$ is a finite metric on $[x]_d$.

**Definition 1.2.13.** A map $f:(X,d_X)\to(Y,d_Y)$ between two metric spaces is called **distance-preserving** if

$$d_Y(f(x_1),f(x_2))=d_X(x_1,x_2), \quad x_1,x_2\in X. \tag{1.2.24}$$

A bijective distance-preserving map is called an **isometry**, Two metric spaces are **isometric** if there is an isometry between them.

**Example 1.2.14.** (1) For any given nonempty set $X$, we can define the **trivial metric**

$$d_{\mathbb{R}}(x,y) := \begin{cases} 0, & x=y, \\ 1 & x\ne y. \end{cases} \tag{1.2.25}$$

(2) Let $X=\mathbb{R}$. There are two useful metrics:

$$d(x,y):=|x-y|, \quad d_{\ln}(x,y):=\ln(1+|x-y|). \tag{1.2.26}$$

The second one appears in the complex algebraic geometry and differential geometry.

(3) Given two metric spaces $(X,d_X)$ and $(Y,d_Y)$, define the **product metric** on $X\times Y$ by

$$d_{X\times Y}((x_1,y_1),(x_2,y_2)) := (d_X(x_1,x_2)+d_Y(y_1,y_2))^{1/2}. \tag{1.2.27}$$

Figure: Felix Hausdorff (1868/11/8 - 1942/1/26)

(4) $X = \mathbb{R}^n$:

$$d_{\mathbb{R}^n}(\boldsymbol{x}, \boldsymbol{y}) := \left( \sum_{1 \le i \le n} (x^i - y^i)^2 \right)^{1/2}.$$

(5) For a metric space $(X, d)$ and $\lambda > 0$, define

$$d_\lambda(x, y) := \lambda d(x, y). \tag{1.2.28}$$

(6) If $(X, d)$ is a metric space and $Y \subseteq X$, we see that $(Y, d_Y := d|_X)$ is itself a metric space.

Assume that $(X, d)$ is a metric space.

(1) We say that $(x_n)_{n \in \mathbb{N}}$ is a **Cauchy sequence** if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for any $\epsilon > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ whenever $n, m \ge n_0$.
(2) $(X, d)$ is said to be **complete** if any Cauchy sequence has a limit in $X$. It is clear that this limit is unique.
(3) $(\mathbb{R} \setminus 0, d_{\mathbb{R}}|_{\mathbb{R} \setminus 0})$ is non-complete.
(4) For $\delta > 0$, define the **$\delta$-neighborhood** of $A \subseteq X$ to be

$$A_\delta := \{x \in X : d(x, A) < \delta\}$$

where $d(x, A) := \inf\{d(x, a) : a \in A\}$.
(5) The **Hausdorff distance** between two given subsets $A, B \subseteq X$ is

$$d_{\mathbf{H}}^X(A, B) := \inf\{\delta > 0 : A \subseteq B_\delta \text{ and } B \subseteq A_\delta\}. \tag{1.2.29}$$

Given metric spaces $(X, d_X)$ and $(Y, d_Y)$, define the **Gromov-Hausdoff distance**

(1.2.30)

$$d_{\mathbf{GH}}((X, d_X), (Y, d_Y)) := \inf \left\{ d_{\mathbf{H}}^Z(f(X), g(Y)) : \begin{array}{c} (Z, d_Z) \text{ metric space and} \\ f : X \hookrightarrow Z,\ g : Y \hookrightarrow Z \\ \text{isometric embeddings} \end{array} \right\},$$

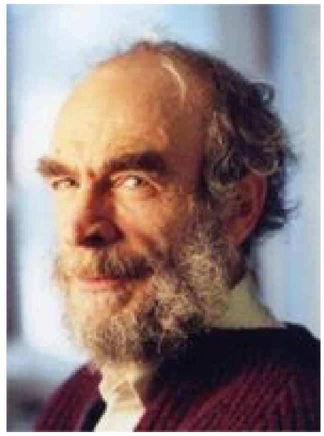

Figure: Mikhail Gromov (1943/12/23 - )

where isometric embeddings mean that $(X, d_X) \to (f(X), d_Z|_{f(X)})$ and $(Y, d_Y) \to (g(Y), d_Z|_{g(Y)})$ are isometries.

We say a sequence of metric spaces $\{(X_n, d_n)\}_{n\geq 1}$ **converges in the Gromov-Hausdorff sense** to a metric space $(X, d)$, written as $(X_n, d_n) \to_{\mathbf{GH}} (X, d)$, if

$$\lim_{n\to\infty} d_{\mathbf{GH}}((x_n, d_n), (X, d)) = 0. \tag{1.2.31}$$

For example, a sequence of cylinders with decreasing to zero radius converges in the Gromov-Hausdorff sense to a line.

This concept is an important tool to study the behavior of "singular space", particularly the study of the Ricci flow (introduced by Hamilton) which leads to a proof (Perelman) of **Poincaré's conjecture** (that is, any closed, simply-connected, three dimensional manifold is diffeomorphic to $\mathbb{S}^3$).

**1.2.5. Functionals.** Consider the function

$$f(x) := x^2, \quad x \in \mathbb{R}.$$

It is easy to check that $f$ is continuous, $\min_{x\in\mathbb{R}} f(x) = f(0) = 0$, and $f'(0) = 0$.

Let $X$ denote the set of all functions defined on $\mathbb{R}$ and consider

$$\mathscr{F} : X \longrightarrow \mathbb{R}, \quad f \longmapsto f(0)^2.$$

Clearly that $\min_{f\in X} \mathscr{F}(f) = \mathscr{F}(0) = 0$.

**Question: How can we define the "derivative" of $\mathscr{F}$?**

**Definition 1.2.15.** A **vector space** (over $\mathbb{R}$) is a set $X$, of elements $x, y, z, \cdots$ (**vectors**), together with two operations of addition $(+)$ and multiplication $(\cdot)$, satisfying

(1) $x + y \in X, \forall\, x, y \in X$,
(2) $a \in \mathbb{R}, x \in X \Longrightarrow a \cdot x \in X$,
(3) $x, y \in X \Longrightarrow x + y = y + x$,
(4) $x, y, z \in X \Longrightarrow (x + y) + z = x + (y + z)$,
(5) $\exists\, 0 \in X$ (**zero vector**) such that $x + 0 = x, \forall\, x \in X$,

(6) $\forall x \in X, \exists -x \in X$ such that $x + (-x) = 0$,
(7) $\forall\, a, b \in \mathbb{R}, \forall\, x \in X \Longrightarrow a \cdot (b \cdot x) = (ab) \cdot x$,
(8) $\forall\, a \in \mathbb{R}, \forall x, y \in X \Longrightarrow a \cdot (x + y) = a \cdot x + a \cdot y$,
(9) $\forall a, b \in \mathbb{R}, \forall x \in X \Longrightarrow (a + b) \cdot x = a \cdot x + b \cdot x$,
(10) $\forall x \in X, 1 \cdot x = x$.

Equivalently, $(X, +, \cdot)$ is a vector space if $(X, +)$ is an Abelian group and $(X, +, \cdot)$ is a left $\mathbb{R}$-module.

**Example 1.2.16.** (1) $\mathbb{R}^n$ is a vector space.

(2) Fix an interval $I \subset \mathbb{R}$, define

$$X := \{\text{real-valued functions defined on } I\}.$$

Let

$$(\phi + \psi)(x) := \phi(x) + \psi(x), \quad (a \cdot \phi)(x) := a \cdot \phi(x).$$

Then $(X, +, \cdot)$ is a vector space.

(3) Let

$$X' := \{f \in X : f(0) - f(1) = 1\}$$

where $X$ is the vector space given in (2) with $I = [0, 1]$. Then $(X', +, \cdot)$ is not a vector space (**Hint:** consider functions $f(x) = 1 - x$ and $g(x) = 1 - x^2$).

**Definition 1.2.17.** A **functional** is a map $\mathscr{F}$ from a vector space $X$ to $\mathbb{R}$.

**Example 1.2.18.** **(Examples of functionals)** (1) $\mathscr{F}(\boldsymbol{x}) := (x^2)^2 - (x^1)^2$ for $\boldsymbol{x} = (x^1, x^2) \in \mathbb{R}^2$.

(2) $X = C[0, \pi/2]$ the space of all continuous functions over $[0, \pi/2]$, and

$$\mathscr{F}(\phi) := \int_0^{\pi/2} \left[2\phi(x)^3 + 9(\sin x)\phi(x)^2 + 12(\sin^2 x)\phi(x) - \cos x\right] dx.$$

(3) $X = \mathbb{R}^2$, and

$$\mathscr{F}(\boldsymbol{x}) := \begin{cases} \frac{xy^2}{x^2+y^4}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

**Definition 1.2.19.** Consider a functional $\mathscr{F} : X \to \mathbb{R}$. The **Gâteaux variation of $\mathscr{F}$ at** $x \in D \subseteq X$ is

(1.2.32)
$$\eth\mathscr{F}(x; h) := \lim_{\epsilon \to 0} \frac{\mathscr{F}(x + \epsilon h) - \mathscr{F}(x)}{\epsilon}.$$

**Example 1.2.20.** (1) $\mathscr{F}(\boldsymbol{x}) := (x^2)^2 - (x^1)^2$ for $\boldsymbol{x} = (x^1, x^2)$,

$$\eth\mathscr{F}(\boldsymbol{x}; \boldsymbol{h}) = \lim_{\epsilon \to 0} \frac{[(x^2 + \epsilon h^2)^2 - (x^1 + \epsilon h^1)^2] - (x^2)^2 - (x^1)^2}{\epsilon} = 2(x^2 h^2 - x^1 h^1).$$

(2) For $\mathscr{F}$ defined in Example 1.2.18 (2),

$$\eth\mathscr{F}(\phi; \phi) = \int_0^{\pi/2} \left[6\phi(x)^2\psi(x) + 18 \sin x\phi(x)\psi(x) + 12 \sin^2 x\psi(x)\right] dx.$$

(3) For $\mathscr{F}$ defined in Example 1.2.18 (3),

$$\eth\mathscr{F}(\boldsymbol{0}; \boldsymbol{h}) = \begin{cases} (h^2)^2/(h^1)^2, & h^1 \neq 0, \\ 0, & h^1 = 0. \end{cases}$$

## 1.3. References


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CHAPTER 2

# Sequences

## 2.1. Convergent sequences

**2.1.1. Definition.** Let $X = \mathbb{R}$, $d(x, y) := |x - y|$. Then $(X, d)$ is a metric space. Actually $(X, d)$ is a complete metric space, i.e., any Cauchy sequence has a limit in $X$. Recall that

$$(x_n)_{n\in\mathbb{N}} \text{ is Cauchy} \iff \forall\epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } d(x_n, x_m) < \epsilon,\ \forall n, m \geq N,$$
$$\lim_{n\to\infty} x_n = x \in X \iff \forall\epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } d(x_n, x_m) < \epsilon,\ \forall n \geq N.$$

**Definition 2.1.1.** Given a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$.

(1) $a \in \mathbb{R}$ is called a **limit** of $(a_n)_{n\in\mathbb{N}}$ if $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon, \quad \text{whenever } n > N.$$

We write $a_n \to a$ or $\lim_{n\to\infty} a_n = a$.

We shall prove that a limit of $(a_n)_{n\in\mathbb{N}}$, if exists, is unique. Hence we can say that $a$ is the limit of $(a_n)_{n\in\mathbb{N}}$.

(2) $(a_n)_{n\in\mathbb{N}}$ is **convergent** if $\exists\ a \in \mathbb{R}$ such that $a_n \to a$. Otherwise, we say that $(a_n)_{n\in\mathbb{N}}$ is **divergent**.

**2.1.2. Examples.** We give some examples to practice "$\epsilon$-$N$".

**Example 2.1.2.** (1) $|q| < 1 \Longrightarrow \lim_{n\to\infty} q^n = 0$.
(2) $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = 0$.
(3) $a \geq 1 \Longrightarrow \lim_{n\to\infty} \sqrt[n]{a} = 1$.
(4) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

PROOF. (1) When $q = 0$, each $q^n$ is zero. Assume now that $0 < |q| < 1$.

$$|q^n - 0| < \epsilon \Longleftrightarrow |q|^n < \epsilon \Longleftrightarrow n > \frac{\ln\epsilon}{\ln|q|}.$$

Hence $\forall\epsilon > 0$, $\exists\ N = \lfloor \ln\epsilon / |\ln|q|| \rfloor$ such that

$$|q^n - 0| < \epsilon \quad \text{whenever } n > N.$$

(2) Write

$$\sqrt{n+1} - \sqrt{n} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}.$$

Then we can take $N = \lfloor 1/4\epsilon^2 \rfloor$.

(3) WLOG, we may assume that $a > 1$. Hence $\sqrt[n]{a} > 1$ and write it as $\sqrt[n]{a} = 1 + y_n$. Because $y_n > 0$ and

$$a = (1+y_n)^n = 1 + ny_n + \frac{n(n-1)}{2}y_n^2 + \cdots + y_n^n > 1 + ny_n,$$

we see that

$$\left|\sqrt[n]{a} - 1\right| = |y_n| < \frac{a-1}{n} \longrightarrow 0.$$

(4) As in (3), write $\sqrt[n]{n} = 1 + y_n$ with $y_n > 0$. Then

$$n = (1+y_n)^n = 1 + ny_n + \frac{n(n-1)}{2}y_n^2 + \cdots + y_n^n > 1 + \frac{n(n-1)}{2}y_n^2$$

and

$$\left|\sqrt[n]{n} - 1\right| = |y_n| < \sqrt{\frac{2}{n-1}} \longrightarrow 0.$$

□

Let $(a_n)_{n\geq 1}$ be a divergent sequence. Then $\forall a \in \mathbb{R}$, $a_n \nrightarrow a$. Thus

$$a_n \nrightarrow a \iff \exists \epsilon_0 > 0,\ \forall N \in \mathbb{N},\ \exists n_0 > N \text{ such that } |a_{n_0} - a| \geq \epsilon_0.$$

**Example 2.1.3.** (1) $\{(-1)^{n-1}\}_{n\geq 1}$ is divergent.

(2) $\{\sin n\}_{n\geq 1}$ is divergent.

PROOF. (1) We first prove that $(-1)^{n-1} \nrightarrow 1$. $\exists \epsilon_0 = 1, \forall N \in \mathbb{N}, \exists n_0 = 2N > N$ such that

$$|a_{n_0} - a| = \left|(-1)^{2n-1} - 1\right| = |-2| = 2 > 1 = \epsilon_0.$$

Next, for any $a \neq 1$, we show that $(-1)^{n-1} \nrightarrow a$. $\exists \epsilon_0 = |a-1|/2$, $\forall N \in \mathbb{N}$, $\exists n_0 = 2N+1$ such that $|a_{n_0} - a| = |1-a| > \epsilon_0$.

(2) Because $|\sin n| \leq 1$, we suffice to show that $\forall\ A \in [-1,1]$, $\sin n \nrightarrow A$. WLOG, we may assume that $0 \leq A \leq 1$. $\exists \epsilon_0 = \sqrt{2}/2, \forall N \in \mathbb{N}, \exists n_0 = \lfloor (2N\pi - \frac{\pi}{2}) + \frac{\pi}{4} \rfloor$ such that $\sin n_0 < -\sqrt{2}/2$ and $|\sin n_0 - A| \geq \sqrt{2}/2 = \epsilon_0$. □

**Remark 2.1.4.** (1) Example 2.1.2 (2) implies that

$$\lim_{n\to\infty}(a_n - a_{n-1}) = 0 \quad \not\Rightarrow \quad \lim_{n\to\infty} a_n \text{ convergent.}$$

(2) Example 2.1.3 implies that

$$\{a_n\}_{n\geq 1} \text{ bounded} \quad \not\Rightarrow \quad \lim_{n\to\infty} a_n \text{ exists.}$$

**Example 2.1.5.** If $\lim_{n\to\infty} a_n = a$, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = a.$$

PROOF. $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon/2$ whenever $n > N_0$. Write

$$\begin{aligned} \left|\frac{a_1 + \cdots + a_n}{n} - a\right| &= \left|\frac{a_1 + \cdots + a_n - na}{n}\right| \\ &= \left|\frac{a_1 + \cdots + a_{N_0} - N_0 a}{n} + \frac{(a_{N_0+1} - a) + \cdots + (a_N - a)}{n}\right| \\ &\leq \frac{|a_1 + \cdots + a_{N_0} - N_0 a|}{n} + \frac{|a_{N_0+1} - a| + \cdots + |a_n - a|}{n} \\ &\leq \frac{n - N_0}{n}\frac{\epsilon}{2} + \frac{|a_1 + \cdots + a_{N_0} - N_0 a|}{n}. \end{aligned}$$

Take

$$N > \max\left\{N_0, \frac{|a_1 + \cdots + a_{N_0} - N_0 a|}{\epsilon/2}\right\}.$$

We then get

$$\left|\frac{a_1 + \cdots + a_n}{n} - a\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

**Example 2.1.6.** Any real number is a limit of some sequence of rational numbers.

PROOF. Given a real number $a \in \mathbb{R}$. Define $a_n := \lfloor na \rfloor / n$. Because

$$na - 1 < \lfloor na \rfloor na,$$

we have

$$a - \frac{1}{n} < \frac{\lfloor na \rfloor}{n} < a$$

or

$$|a_n - a| = \left|\frac{\lfloor na \rfloor}{n} - a\right| < \frac{1}{n} \longrightarrow 0.$$

□

## 2.2. Properties of convergent sequences

**2.2.1. Basic properties.** We left a question in Definition 2.1.1 that a limit, if exists, is unique. In this subsection we shall prove this fact.

**Theorem 2.2.1.** *(1)* $a_n \to a, a_n \to b \Longrightarrow a = b$.
*(2)* $\{a_n\}_{n\geq 1}$ *is convergent* $\Longrightarrow \{a_n\}_{n\geq 1}$ *is bounded.*
*(3)* $a_n \to a$, $b_n \to b$, $a < b \Longrightarrow \exists N \in \mathbb{N}$ *such that* $a_n < b_n$, $\forall n \geq N$.
*(4)* $a_n \to a$ *and* $b < a < c \Longrightarrow \exists N \in \mathbb{N}$ *such that* $b < a_n < c$, $\forall n > N$.
*(5)* $a_n \to a$, $b_n \to b$, $a_n \leq b_n$ $(\forall n > N) \Longrightarrow a \leq b$.
*(6)* $a_n \to a \Longrightarrow |a_n| \to |a|$.

PROOF. (1) Given $\epsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon \quad (\forall n > N_1) \quad \text{and} \quad |b_n - b| < \epsilon \quad (\forall n > N_2).$$

Then

$$|a - b| \leq |a_n - b + |a_n - a| < 2\epsilon \quad (\forall n > \max(N_1, N_2)).$$

By the arbitrary of $\epsilon$ we must have $a = b$.

(2) Take $\epsilon = 1$, $\exists N_1 \in \mathbb{N}$ such that $a - 1 < a_n < a + 1, \forall n > N_1$. Hence $\forall n \geq 1$,

$$\min\{a_1, \cdots, a_N, a - 1\} \leq a_n \leq \max\{a_1, \cdots, a_N, a + 1\}.$$

(3) Take $\epsilon = \frac{b-a}{2} > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{b-a}{2} \quad (n > N_1) \quad \text{and} \quad |b_n - b| < \frac{b-a}{2} \quad (n > N_2).$$

Hence

$$a_n < \frac{b-a}{2} + a = \frac{b+a}{2} < b_n \quad (n > \max(N_1, N_2)).$$

(4) Letting $b_n >= b$ in (3), we can find $N \in \mathbb{N}$ such that $b = b_n < a_n$ $(n > N)$.

(5) If $\lim_{n\to\infty} b_n = b < a = \lim_{n\to\infty} a_n$, then by (3), we have $b_n < a_n$ for al $n > N$.

(6) $a_n \to a$ implies that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$. Hence $||a_n| - |a|| \leq |a_n - a| < \epsilon$. □

**Remark 2.2.2.** (1) $\{a_n\}_{n\geq 1}$ is bounded $\not\Rightarrow$ $\{a_n\}_{n\geq 1}$ is convergent.

(2) $a_n \to a$, $b_n \to b$, $a_n < b_n \not\Rightarrow a < b$. For example, $a_n = 1/n$, $b_n = 2/n$, but $a = b = 0$.

(3) $\{|a_n|\}_{n\geq 1}$ is convergent $\not\Rightarrow$ $\{a_n\}_{n\geq 1}$ is convergent.

**Theorem 2.2.3.** *If $x_n \leq y_n \leq z_n$ holds for any $n \geq N_0$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$, then $\lim_{n\to\infty} y_n = a$.*

PROOF. $\forall \epsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - a| < \epsilon, \quad |z_n - a| < \epsilon.$$

Then

$$a - \epsilon < x_n \leq y_n \leq z_n < a + \epsilon$$

for all $n > \max(N_0, N_1, N_2)$. Hence $x_n \to a$. □

**Example 2.2.4.** (1) $a_1, \cdots, a_k > 0 \Longrightarrow$

$$\lim_{n\to\infty} \sqrt[n]{a_1^n + \cdots + a_k^n} = \max\{a_1, \cdots, a_k\}. \tag{2.2.1}$$

WLOG, we may assume that $\max\{a_1, \cdots, a_k\} = a_1$. Then

$$a_1 < \sqrt[n]{a_1^n + \cdots + a_k^n} \leq \sqrt[n]{k a_1^n} = (\sqrt[n]{k}) a_1 \to a_1.$$

(2) One has

$$\lim_{n\to\infty} \frac{1+\sqrt[n]{2}+\cdots+\sqrt[n]{n}}{n} = 1. \tag{2.2.2}$$

Indeed,

$$\frac{1+1+\cdots+1}{n} \le \frac{1+\sqrt[n]{2}+\cdots+\sqrt[n]{n}}{n} \le \frac{\sqrt[n]{n}+\cdots+\sqrt[n]{n}}{n}$$

so $1 \le (1+\sqrt[n]{2}+\cdots+\sqrt[n]{n})/n \le \sqrt[n]{n} \to 1$.

**2.2.2. Algebraic operations.** Suppose we have two sequences $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$, we can ask the behaviors of $a_n \pm b_n$, $a_nb_n$, and $a_n/b_n$ ($b_n \ne 0$ for larger $n$).

**Theorem 2.2.5.** *Assume that $a_n \to a$, $b_n \to b$, and $\alpha, \beta \in \mathbb{R}$. Then*

$$\alpha a_n \pm \beta b_n \to \alpha a \pm \beta b, \quad a_nb_n \to ab, \quad \frac{a_n}{b_n} \to \frac{a}{b}\ (b \ne 0). \tag{2.2.3}$$

PROOF. (1) $b_b \to b$ implies $-b_n \to -b$. We may prove $\alpha a_n + \beta b_n \to \alpha a + \beta b$.

$$0 \le |(\alpha a_n + \beta b_n) - (\alpha a + \beta b)| \le |\alpha||a_n - a| + |\beta||b_n - b| \to 0.$$

(2) $\{a_n\}, \{b_n\}$ are convergent $\Longrightarrow |a_n| \le M_1$ and $|b_n| \le M_2$.

$$0 \le |a_nb_n - ab| = |a_n(b_n - b) + (a_n - a)b| \le M_1|b_n - b| + M_2|a_n - a| \to 0.$$

(3) $b_n \to b \Longrightarrow |b_n| \to |b|$. Since $|b| > 0$, it follows that $|b_n| > |b|/2$ for $n \gg 1$.

$$0 \le \left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{b(a_n - a) - a(b_n - b)}{b_nb}\right|$$

$$\le \frac{|b||a_n - a| + |a||b_n - b|}{|b_n||b|} \le \frac{2}{|b|^2}\left(|b||a_n - a| + |a||b_n - b|\right) \to 0.$$

□

**Remark 2.2.6.** (1) $\lim_{n\to\infty}(a_n + b_n)$ exists $\not\Rightarrow$ $\lim_{n\to\infty} a_n$ exists or $\lim_{n\to\infty} b_n$ exists. For example, $a_n = (-1)^{n-1}$, $b_n = (-1)^n$.

(2) $\lim_{n\to\infty} a_nb_n$ exists $\not\Rightarrow$ $\lim_{n\to\infty} a_n$ exists or $\lim_{n\to\infty} b_n$ exists. For example, $a_n = b_n = (-1)^{n-1}$.

(3) $\lim_{n\to\infty} a_n/b_n$ exists $\not\Rightarrow$ $\lim_{n\to\infty} a_n$ exists or $\lim_{n\to\infty} b_n$ exists. For example, $a_n = (-1)^n$, $b_n = n$.

**Example 2.2.7.** (1) For all $a > 0$,

$$\lim_{n\to\infty} \sqrt[n]{a} = 1. \tag{2.2.4}$$

Example 2.1.2 implies (2.2.4) holds for $a \geq 1$. When $0 < a < 1$,

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{1}{a}}} = \frac{1}{1} = 1.$$

(2) For $q > 1$,

$$\lim_{n\to\infty} \frac{\log_q n}{n} = 0. \tag{2.2.5}$$

In fact,

$$\lim_{n\to\infty} \sqrt[n]{n} < q^{\epsilon} \quad \Longrightarrow \quad \sqrt[n]{n} < q^{\epsilon} \ \ (\forall n > N).$$

So

$$\frac{\log_q n}{n} < \epsilon \ (\forall n > N) \quad \Longrightarrow \quad \frac{\log_q n}{n} \to 0.$$

**2.2.3. Infinitely small and infinitely large sequences.** A sequence $\{a_n\}_{n\geq 1}$ is said to be an **infinitely small sequence**, if $\lim_{n\to\infty} a_n = 0$ or $a_n \to 0$.

(1) $a_n \to a \Longleftrightarrow a_n - a \to 0 \Longleftrightarrow a_n = a + \alpha_n$ with $\alpha_n \to 0$.
(2) $a_n \to 0 \Longleftrightarrow |a_n| \to 0$.
(3) $a_n \to 0$, $b_n \to 0 \Longrightarrow a_n + b_n, a_n - b_n, a_n b_n \to 0$.
(4) $a_n \to 0$, $b_n \to 0 \not\Rightarrow a_n/b_n \to 0$. For example,

$$\begin{cases} a_n = \frac{1}{n}, \\ b_n = \frac{1}{n}, \end{cases} \quad \frac{a_n}{b_n} \equiv 1, \quad \begin{cases} a_n = \frac{1}{n}, \\ b_n = \frac{1}{n^2}, \end{cases} \quad \frac{a_n}{b_n} = n, \quad \begin{cases} a_n = \frac{1}{n^2}, \\ b_n = \frac{1}{n}, \end{cases} \quad \frac{a_n}{b_n} = \frac{1}{n}.$$

(5) $a_n \to 0$, $|b_n| \leq M \Longrightarrow a_n b_n \to 0$.

A sequence $\{a_n\}_{n\geq 1}$ is said to be an **infinitely large sequence**, if $\forall C > 0$, $\exists N \in \mathbb{N}$ such that

$$|a_n| \geq C \quad \text{whenever} \quad n > N.$$

Notation: $\lim_{n\to\infty} a_n = \infty$ or $a_n \to \infty$.

(1) Write

$$\begin{matrix} \lim_{n\to\infty} a_n = +\infty \\ \text{or } a_n \to +\infty \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} \{a_n\}_{n\geq 1} \text{ is an infinitely large sequence} \\ \text{and } a_n > 0 \ (\forall n \geq N_0). \end{matrix}$$

(2) Write

$$\begin{matrix} \lim_{n\to\infty} a_n = -\infty \\ \text{or } a_n \to -\infty \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} \{a_n\}_{n\geq 1} \text{ is an infinitely large sequence} \\ \text{and } a_n < 0 \ (\forall n \geq N_0). \end{matrix}$$

(3) $a_n \to +\infty$ or $a_n \to -\infty \Longrightarrow a_n \to \infty$. But the convergence is not true, for example $a_n = (-1)^n n$.
(4) $a_n, b_n \to \pm\infty \Longrightarrow a_n + b_n \to \pm\infty$.
(5) $a_n \to \pm\infty$, $b_n \to \mp\infty \Longrightarrow a_n - b_n \to \pm\infty$.
(6) $a_n \to \infty$, $|b_n| \leq M \Longrightarrow a_n b_n \to \infty$.
(7) $a_n, b_n \to \pm\infty \Longrightarrow a_n b_n \to \pm\infty$.
(8) $a_n \to \pm\infty$, $b_n \to \mp\infty \Longrightarrow a_n b_n \to -\infty$.
(9) $a_n \to 0$, $a_n \neq 0 \Longrightarrow 1/a_n \to \infty$.

**Example 2.2.8.** (1) $|q| > 1 \Longrightarrow q^n \to \infty$. Indeed,

$$|q^n| = |q|^n \geq |q|^{\frac{\ln C}{\ln |q|}} = C$$

for $n \geq N = \lfloor \ln C / \ln |q| \rfloor$.

(2) $a_n := \sum_{1 \leq k \leq n} 1/(\sqrt{n} + \sqrt{k}) \to +\infty$. Indeed,

$$a_n > \frac{n}{2\sqrt{n}} = \frac{\sqrt{n}}{2} \to +\infty.$$

(3) Set

$$a_n := \frac{x_0 n^k + x_1 n^{k-1} + \cdots + x_{k-1} n + x_k}{y_0 n^\ell + y_1 n^{\ell-1} + \cdots + y_{\ell-1} n + y_\ell}, \quad (k, \ell \in \mathbb{N},\ x_0 y_0 \neq 0).$$

Since

$$a_n = n^{k-\ell} \frac{x_0 + \frac{x_1}{n} + \cdots + \frac{x_{k-1}}{n^{k-1}} + \frac{x_k}{n^k}}{y_0 + \frac{y_1}{n} + \cdots + \frac{y_{\ell-1}}{n^{\ell-1}} + \frac{y_\ell}{n^\ell}},$$

we get

$$\lim_{n\to\infty} a_n = \begin{cases} 0, & k < \ell, \\ x_0/y_0, & k = \ell, \\ \infty, & k > \ell. \end{cases}$$

(4) $a_n = \sqrt[n]{n!} \to +\infty$. Observe that

$$(n!)^2 = (1 \cdot n)[2 \cdot (n-1)] \cdots [k(n-k)] \cdot (n \cdot 1) = \prod_{1 \leq k \leq n} [k(n-k+1)] \geq n^2$$

since $k(n-k+1) \geq n$ which can be deduced from the inequality $(k-1)(n-k) \geq 0$ $(1 \leq k \leq n)$.

**2.2.4. Stolz's theorems.** These theorems are used to deal with "$\infty/\infty$ or "$0/0$" limits.

**Theorem 2.2.9. (Stolz's theorem I: "$\infty/\infty$" type)** *Given two sequence $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$. If*

$$y_n < y_{n+1}, \quad y_n \to +\infty, \quad \text{and} \quad \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = a \ (\text{a real number or } \pm\infty),$$

*then*

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = a. \tag{2.2.6}$$

PROOF. **Case 1:** $a = 0$. $\forall \epsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that

$$|x_n - x_{n-1}| < \epsilon(y_n - y_{n-1}), \quad \forall\, n > N_1.$$

In particular,

$$|x_n - x_{N_1}| \leq \sum_{N_1+1 \leq i \leq n} |x_i - x_{i-1}| \leq \sum_{N_1+1 \leq i \leq n} \epsilon(y_i - y_{i-1}) = \epsilon(y_n - y_{N_1});$$

thus

$$\left|\frac{x_n}{y_n} - \frac{x_{N_1}}{y_n}\right| \le \epsilon\left(1 - \frac{y_{N_1}}{y_n}\right) < \epsilon.$$

But $y_n \to +\infty$, $\exists N_2 \in \mathbb{N}$ such that $|x_{N_1}/y_n| < \epsilon$ whenever $n > N_2$. Finally,

$$\left|\frac{x_n}{y_n}\right| \le \epsilon + \epsilon = 2\epsilon$$

whenever $n > N = \max\{N_1, N_2\}$.

**Case 2:** $a \neq 0$**.** The basic idea is to construct new sequences and then apply **Case 1**. Let

$$\widetilde{x}_n := x_n - ay_n.$$

Then

$$\frac{\widetilde{x}_n - \widetilde{x}_{n-1}}{y_n - y_{n-1}} = \frac{(x_n - ay_n) - (x_{n-1} - ay_{n-1})}{y_n - y_{n-1}} = \frac{x_n - x_{n-1}}{y_n - y_{n-1}} - a \to 0.$$

From **Case 1**, we have $\widetilde{x}_n/y_n \to 0$ or $x_n/y_0 \to a$.

**Case 3:** $a = +\infty$**.** $\exists N_1 \in \mathbb{N}$ such that $x_n - x_{n-1} > y_n - y_{n-1} > 0$ $(\forall n > N_1)$. Moreover

$$x_n - x_{N_1} = \sum_{N_1+i\le i\le n}(x_i - x_{i-1}) > \sum_{N_1+1\le i\le n}(y_i - y_{i-1}) = y_n - y_{N_1}.$$

Letting $n \to +\infty$ yields $x_n \to +\infty$. According to **Case 1**,

$$\lim_{n\to\infty}\frac{y_n}{x_n} = \lim_{n\to\infty}\frac{y_n - y_{n-1}}{x_n - x_{n-1}} = \frac{1}{+\infty} = 0.$$

**Case 4:** $a = -\infty$**.** Observe that

$$\frac{x_n - x_{n-1}}{y_n - y_{n-1}} \to -\infty \Longleftrightarrow \frac{(-x_n) - (-x_{n-1})}{y_n - y_{n-1}} \to +\infty.$$

Now the last case follows from **Case 3**. □

**Theorem 2.2.10. (Stolz's theorem II: "0/0" type)** *Given two sequences* $\{x_n\}_{n\ge1}$ *and* $\{y_n\}_{n\ge1}$*. If*

$$x_n \to 0, \quad y_n > y_{n+1}, \quad y_n \to 0, \quad \text{and} \quad \lim_{n\to\infty}\frac{x_n - x_{n+1}}{y_n - y_{n+1}} = a \ (a \text{ real number or } \pm\infty)$$

*then*

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{x_n - x_{n+1}}{y_n - y_{n+1}} = a. \tag{2.2.7}$$

PROOF. **Case 1:** $a \in \mathbb{R}$**.** $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such thst

$$a - \epsilon < \frac{x_n - x_{n+1}}{y_n - y_{n+}} < a + \epsilon, \quad \forall n > N,$$

or

$$(a - \epsilon)(y_n - y_{n+1}) < x_n - x_{n+1} < (a + \epsilon)(y_n - y_{n+1}), \quad \forall n > N.$$

In particular,

$$(a-\epsilon)(y_n - y_{n+p}) < x_n - x_{n+p} < (a+\epsilon)(y_n - y_{n+p}), \quad \forall n > N \text{ and } p \geq 1.$$

Letting $p \to +\infty$ yields

$$(a-\epsilon)y_n \leq x_n \leq (a+\epsilon)y_n \Longrightarrow \left|\frac{x_n}{y_n} - a\right| \leq \epsilon.$$

**Case 2:** $a = +\infty$. Given $C > 0$, $\exists N \in \mathbb{N}$ such that $x_n - x_{n+1} > C(y_n - y_{n+1}) \Longrightarrow x_n - x_{n+p} > C(y_n - y_{n+p})$, $\forall n > N$ and $p \geq 1$. Letting $p \to +\infty$ yields $x_n/y_n \geq C$.

**Case 3:** $a = -\infty$. The proof is similar to that given in Theorem 2.2.9 **Case 4.** □

**Example 2.2.11.** (1) $a_n \to a \Longrightarrow$

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n^2} = \frac{a}{2}.$$

(2) $\lim_{n\to\infty} a_n = a$ or $\pm\infty \Longrightarrow \lim_{n\to\infty} \frac{1}{n}(a_1 + \cdots + a_n) = a$.

(3) $\lim_{n\to\infty}(a_{n+1} - a_n) = \ell \Longrightarrow$ Find $\lim_{n\to\infty} a_n/n$ and $\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\leq i\leq n} a_i$.

(4) $a_n \leq a_{n+1}$, $\lim_{n\to\infty} \frac{1}{n}(a_1 + \cdots + a_n) = a \Longrightarrow \lim_{n\to\infty} a_n = a$.

(5) $a_n = s_n - s_{n-1}$, $\sigma_n = \frac{1}{n+1}(s_0 + \cdots + s_n)$, $na_n \to 0$, $\sigma_n$ is convergent $\Longrightarrow s_n$ is also convergent and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sigma_n.$$

PROOF. (1) Let $y_n := a_1 + 2a_2 + \cdots + na_n$, and $y_n := n^2$. By Theorem 2.2.9,

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} \frac{na_n}{n^2 - (n-1)^2} = \lim_{n\to\infty} \frac{na_n}{2n-1} = \frac{a}{2}.$$

(2) Let $x_n = a_1 + \cdots + a_n$ and $y_n$. Then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} a_n = a.$$

(3) Let $x_n := a_{n+1} - a_n$ and

$$y_n := \frac{1}{n+1} \sum_{1\leq i\leq n} x_i = \frac{a_{n+1} - a_0}{n+1}.$$

Because $x_n \to \ell$, we obtain $y_n \to \ell$ (by (2)) and then $a_n/n \to \ell$. Similarly set

$$\widetilde{y}_n := \frac{1}{n^2} \sum_{1\leq i\leq n} i(a_{i+1} - a_i) = \frac{1}{n^2}\left(na_{n+1} - \sum_{1\leq i\leq n} a_i\right).$$

By (1), $\widetilde{y}_n \to \ell/2$ so that $\sum_{1\leq i\leq n} a_i/n^2 = \frac{a_{n+1}}{n} - \widetilde{y}_n \to \ell - \frac{\ell}{2} = \frac{\ell}{2}$.

(4) Since $a_n \leq a_{n+1}$, it follows that $\sigma_n := \frac{1}{n}(a_1 + \cdots + a_n) \leq na_n/n = a_n$. On the other hand, for all $m > n$,

$$\sigma_m = \frac{1}{m}\left(\sum_{1\leq i\leq n} a_i + \sum_{n+1\leq i\leq m} a_i\right) \geq \frac{1}{m} \sum_{1\leq i\leq n} a_i + \frac{m-n}{m} a_n \to a_n$$

as $m \to \infty$. Therefore $\sigma_n \to a$.

(5) Observe that

$$s_n - \sigma_n = \frac{1}{n+1}\sum_{1\le i\le n} ia_i.$$

Then

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{1\le i\le n} ia_i = \lim_{n\to\infty}\frac{\sum_{1\le i\le n} ia_i - \sum_{1\le i\le n-1} ia_i}{(n+1)-n} = \lim_{n\to\infty} na_n = 0.$$

□

(6) According to (3), we know that $a_n \to a \Longrightarrow \frac{1}{n}(a_1+\cdots+a_n)\to a$. However, the converse is not true. For example, consider the sequence $\{(-1)^n\}_{n\ge 1}$.

**Example 2.2.12.** (1) $k\in\mathbb{Z}_+ \Longrightarrow$

$$\lim_{n\to\infty} n\left(\frac{1^k+2^k+\cdots+n^k}{n^{k+1}} - \frac{1}{k+1}\right) = \frac{1}{2}.$$

(2) $\lim_{n\to\infty} n(a_n - a) = b,\ k\in\mathbb{Z}_+ \Longrightarrow$

$$\lim_{n\to\infty} n\left(\frac{a_1+2^k a_2+\cdots+n^k a_n}{n^{k+1}} - \frac{a}{k+1}\right) = \frac{b}{k}+\frac{a}{2}.$$

PROOF. (1) Using Theorem 2.2.9 we have

$$\begin{aligned}
\lim_{n\to\infty} n\left(\frac{1^k+\cdots+n^k}{n^{k+1}} - \frac{1}{k+1}\right) &= \lim_{n\to\infty}\frac{(k+1)(1^k+\cdots+n^k)-n^{k+1}}{(k+1)n^k}\\
= \lim_{n\to\infty}&\frac{[(k+1)(1^k+\cdots+n^k)-n^{k+1}]-[(k+1)(1^k+\cdots+(n-1)^k)-(n-1)^{k+1}]}{(k+1)n^k-(k+1)(n-1)^k}\\
= \lim_{n\to\infty}\frac{(k+1)n^k-[n^{k+1}-(n-1)^{k+1}]}{(k+1)[n^k-(n-1)^k]} &= \lim_{n\to+\infty}\frac{\frac{1}{2}k(k+1)n^k+\cdots}{k(k+1)n^k+\cdots} = \frac{1}{2}.
\end{aligned}$$

(2) Observe that

$$\begin{aligned}
&\lim_{n\to\infty} n\left(\frac{a_1+2^k a_2+\cdots+n^k a_n}{n^{k+1}} - \frac{a}{k+1}\right)\\
= &\lim_{n\to\infty}\frac{(k+1)\sum_{1\le i\le n} i^k a_i - a n^{k+1}}{(k+1)n^k} = \lim_{n\to\infty}\frac{(k+1)n^k a_n - a(n^{k+1}-(n-1)^{k+1})}{(k+1)[n^k-(n-1)^k]}
\end{aligned}$$

and

$$n^{k+1}-(n-1)^{k+1} = (n-1+1)^{k+1}-(n-1)^{k+1} = (k+1)(n-1)^k + \frac{k(k+1)}{2}(n-1)^{k-1}+\cdots.$$

Hence

$$\begin{aligned}
&\lim_{n\to\infty} n\left(\frac{a_1+2^k a_2+\cdots+n^k a_n}{n^{k+1}} - \frac{a}{k+1}\right)\\
= &\lim_{n\to\infty}\frac{(k+1)a_n n^k - (k+1)a(n-1)^k - \frac{ak(k+1)}{2}(n-1)^{k-1}+\cdots}{(k+1)[k(n-1)^{k-1}+\cdots]}
\end{aligned}$$

$$= \lim_{n\to\infty} \frac{(k+1)n^k(a_n-a)+(k+1)a[n^k-(n-1)^k]-\frac{k(k+1)a}{2}(n-1)^{k-1}+\cdots}{(k+1)[k(n-1)^k+\cdots]}$$
$$= \frac{b}{k}+a-\frac{a}{2} = \frac{b}{k}+\frac{a}{2}.$$

□

**Theorem 2.2.13. (Toeplitz's theorem)** *Assume that* $p_{n0}+p_{n1}+\cdots+p_{nn}=1$ *for all* $n\in\mathbb{N}$, *and each* $p_{ij}\geq 0$. *Let*

$$y_n := \sum_{0\leq i\leq n} p_{ni}x_i, \quad n\in\mathbb{N}.$$

*Then TFAE:*

(i) $x_n\to a \Longrightarrow y_n\to a$,
(ii) $p_{nm}\to 0$ *for each* $m\in\mathbb{N}$.

PROOF. $\Longrightarrow$: Take $x_n=\delta_{nm}$. Then $x_n\to 0$ and $y_n=p_{nm}$ $(n\geq m)$. Hence

$$\lim_{n\to\infty} p_{nm} = \lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = 0.$$

$\Longleftarrow$: Suppose $x_n\to a$. Then $\exists M>0$ such that $|x_n-a|\leq M$ for all $n\in\mathbb{Z}_+$. $\forall\epsilon>0$, $\exists N^*\in\mathbb{N}$ such that $|x_n-a|<\epsilon/2$ for all $n>N^*$. But from $\lim_{n\to\infty} p_{ni}=0$, we get that $\exists N_i>N^*$ such that

$$0\leq p_{ni}\leq \frac{\epsilon}{2N^*M}, \quad n>N_i.$$

Let $N:=\max_{0\leq i\leq N^*} N_i$. Then

$$\begin{aligned}|y_n-a| &= \left|\sum_{0\leq i\leq n} p_{ni}x_i - \sum_{0\leq i\leq n} p_{ni}a\right| \\ &\leq \sum_{0\leq i\leq N^*} p_{ni}|x_i-a| + \sum_{N^*+1\leq i\leq n} p_{ni}|x_i-a| \\ &< MN^*\frac{\epsilon}{2N^*M}+\frac{\epsilon}{2}\sum_{N^*+1\leq i\leq n} p_{ni} < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.\end{aligned}$$

□

**Example 2.2.14.** (1) $b_n>0$, $b_0+b_1+\cdots+b_n\to\infty$, $a_n/b_n\to s\Longrightarrow$

$$\lim_{n\to\infty}\frac{a_0+a_1+\cdots+a_n}{b_0+b_1+\cdots+b_n}=s.$$

(2) $p_k>0$, $\frac{p_n}{p_0+p_1+\cdots+p_n}\to 0$, $s_n\to s\Longrightarrow$

$$\lim_{n\to\infty}\frac{\sum_{0\leq i\leq n} s_ip_{n-i}}{\sum_{0\leq i\leq n} p_i}=s.$$

(3) $p_k, q_k > 0$, $\frac{p_n}{p_0+\cdots+p_n} \to 0$, $\frac{q_n}{q_0+\cdots+q_n} \to 0 \Longrightarrow$

$$\lim_{n\to\infty} \frac{r_n}{\sum_{0\le i\le n} r_i} = 0.$$

Here $r_n := \sum_{0\le i\le n} p_i q_{n-i}$.

PROOF. (1) Let $x_n := a_n/b_n$, $p_{nm} := b_m/\sum_{0\le i\le n} b_i$, and $y_n := \sum_{0\le i\le n} p_{ni}x_i$. Then

$$\lim_{n\to\infty} p_{nm} = 0, \quad \sum_{0\le i\le n} p_{ni} = 1, \quad p_{nm} \ge 0.$$

By Theorem 2.2.13,

$$\lim_{n\to\infty} \frac{\sum_{0\le i\le n} a_i}{\sum_{0\le i\le n} b_i} = \lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = s.$$

(2) Let $p_{nm} := p_{n-m}/\sum_{0\le i\le n} p_i$, where $0 \le m \le n$ and $n = 1, 2, \cdots$, and

$$x_n := s_n, \quad y_n := \sum_{0\le i\le n} p_{ni}x_i = \frac{\sum_{0\le i\le n} s_i p_{n-i}}{\sum_{0\le i\le n} p_i}.$$

(3) Let

$$P_n := \sum_{0\le i\le n} p_i, \quad Q_n := \sum_{0\le i\le n} q_n, \quad R_n := \sum_{0\le i\le n} r_i,$$

and

$$p_{nm} := \frac{p_{n-m}Q_m}{\sum_{0\le i\le n} p_i Q_{n-i}}, \quad x_n := \frac{q_n}{Q_n}, \quad y_n := \sum_{0\le i\le n} p_{ni}x_i.$$

□

**Example 2.2.15.** (1) Show that

$$\left(\frac{n}{3}\right)^n < n! < \left(\frac{n+2}{\sqrt{6}}\right)^n. \tag{2.2.8}$$

(2) Show that

$$n < \left(1 + \frac{2}{\sqrt{n}}\right)^n. \tag{2.2.9}$$

PROOF. (1) Recall that

$$(n!)^2 = \prod_{1\le k\le n} [k(n-k+1)] \ge n^n$$

so that

$$n! > n^{n/2} = (\sqrt{n})^n.$$

The inequality (2.2.8) gives a better lower bound for $n!$ than $(\sqrt{n})^n$. Suppose that $k! > (k/3)^k$ holds. Since

$$(k+1)! = (k+1)k! > (k+1)\left(\frac{k}{3}\right)^k,$$

it follows that in order to prove $(k+1)! > ((k+1)/3)^{k+1}$ we shall prove

$$(k+1)\left(\frac{k}{3}\right)^k > \left(\frac{k+1}{3}\right)^{k+1}$$

or

$$3k^k > (k+1)^k \iff \left(1+\frac{1}{k}\right)^k < 3.$$

But

$$\begin{aligned}\left(1+\frac{1}{k}\right)^k &= 1+1+\sum_{2\le i\le k}\frac{k(k-1)\cdots(k-i+1)}{i!}\frac{1}{k^i}\\ &< 2+\sum_{2\le i\le k}\frac{1}{i!} < 2+\sum_{2\le i\le k}\frac{1}{i(i-1)} < 3.\end{aligned}$$

Assume that $k! < ((k+2)/\sqrt{6})^k$. From $(k+1)! = (k+1)k! < (k+1)((k+2)/\sqrt{6})^k$, we shall prove that

$$(k+1)\left(\frac{k+2}{\sqrt{6}}\right)^k < \left(\frac{k+3}{\sqrt{6}}\right)^{k+1}.$$

Indeed,

$$\begin{aligned}\left(\frac{k+3}{\sqrt{6}}\right)^{k+1} &= \left(\frac{k+2}{\sqrt{6}}+\frac{1}{\sqrt{6}}\right)^{k+1} > \left(\frac{k+2}{\sqrt{6}}\right)^{k+1} + (k+1)\left(\frac{k+2}{\sqrt{6}}\right)^k\frac{1}{\sqrt{6}}\\ &+\frac{(k+1)k}{2}\left(\frac{k+2}{\sqrt{6}}\right)^{k-1}\frac{1}{(\sqrt{6})^2}+\frac{(k+1)k(k-1)}{6}\left(\frac{k+2}{\sqrt{6}}\right)^{k-2}\frac{1}{(\sqrt{6})^3}\\ &= \left(\frac{k+2}{\sqrt{6}}\right)^k\left[\frac{k+2}{\sqrt{6}}+\frac{k+1}{\sqrt{6}}+\frac{k(k+1)}{2\sqrt{6}(k+2)}+\frac{(k+1)k(k-1)}{6\sqrt{6}(k+2)^2}\right]\\ &= \left(\frac{k+2}{\sqrt{6}}\right)^k\frac{16k^3+75k^2+125k+72}{6\sqrt{6}(k+2)^2}.\end{aligned}$$

We now claim that

$$\frac{16k^3+75k^2+125k+72}{6\sqrt{6}(k+2)^2} > k+1 \Leftrightarrow 16k^3+75k^2+125k+72 > 6\sqrt{6}(k^3+5k^2+8k+4).$$

But this follows from the following observation: $16 > 6\sqrt{6}$, $75 > 30\sqrt{6}$, $125 > 48\sqrt{6}$, and $72 > 24\sqrt{6}$.

(2) it follows from

$$\left(1+\frac{2}{\sqrt{n}}\right) = 1+n\cdot\frac{2}{\sqrt{n}}+\frac{n(n-1)}{2}\left(\frac{2}{\sqrt{n}}\right)^2+\cdots > \frac{n(n-1)}{2}\frac{4}{n} = 2(n-1).$$

When $n \ge 2$, we get $2(n-1) \ge n$. □

**Example 2.2.16.** Let $x_1 = a$, $x_2 = b$, and $x_n = (x_{n-1}+x_{n-2})/2$. Find $\lim_{n\to\infty} x_n$.

PROOF. Observe that

$$x_{n+1} - x_n = \frac{x_n + x_{n-1}}{2} - x_n = \frac{x_{n-1} - x_n}{2} = \cdots = \frac{x_2 - x_1}{(-2)^{n-1}} = \frac{b-a}{(-2)^{n-1}}$$

and

$$\begin{aligned} x_{n+1} &= \sum_{1 \le m \le n} (x_{m+1} - x_m) + x_1 = (b-a) \sum_{1 \le m \le n} \frac{1}{(-2)^{m-1}} + a \\ &= (b-a)\frac{1-(-1/2)^{n-1}}{1-(-1/2)} + a \to \frac{2}{3}(b-a) + a = \frac{2b+a}{3}. \end{aligned}$$

□

**Example 2.2.17.** (1) Suppose that $\lambda \in \mathbb{R}$ and $|\lambda| < 1$. Then

$$\lim_{n\to\infty} a_n = a \iff \lim_{n\to\infty} (a_{n+1} - \lambda a_n) = (1-\lambda)a.$$

(2) We have

$$\lim_{n\to\infty} a_n = a \iff \lim_{n\to\infty} (4a_{n+2} - 4a_{n+1} + a_n) = a.$$

PROOF. (1) The "⇒" is clear. Now we assume that $a_{n+1} - \lambda a_n \to (1-\lambda)a$. Let

$$x_n := a_{n+1} - \lambda a_n.$$

Then

$$\frac{a_{n+1}}{\lambda^{n+1}} = \frac{a_n}{\lambda^n} + \frac{x_n}{\lambda^{n+1}}$$

and

$$a_n = \lambda^n \left( a_0 + \sum_{1 \le k \le n} \frac{x_{k-1}}{\lambda^k} \right), \quad n \in \mathbb{Z}_+.$$

When $0 < \lambda < 1$, we have $\lambda^n \to 0$ and by Theorem 2.2.9

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{a_0 + \sum_{1\le k\le n} \frac{x_{k-1}}{\lambda^k}}{(\frac{1}{\lambda})^n} = \lim_{n\to\infty} \frac{\frac{x_n}{\lambda^{n+1}}}{(\frac{1}{\lambda})^{n+1} - (\frac{1}{\lambda})^n} \\ &= \lim_{n\to\infty} \frac{x_n}{1-\lambda} = \frac{(1-\lambda)a}{1-\lambda} = a. \end{aligned}$$

When $\lambda = 0$, the conclusion is trivial. When $-1 < \lambda < 0$, we consider the term

$$a_{2n} = \lambda^{2n} \left( a_0 + \sum_{1 \le k \le 2n} \frac{x_{k-1}}{\lambda^k} \right).$$

Hence

$$\lim_{n\to\infty} a_{2n} = \frac{1}{1-\lambda^2} \lim_{n\to\infty} (x_{2n+1} + \lambda x_{2n}) = a.$$

Similarly,

$$-a_{2n+1} = (-\lambda)^{2n+1} \left( a_0 + \sum_{1 \le k \le 2n+1} \frac{x_{k-1}}{\lambda^k} \right)$$

and

$$\lim_{n\to\infty} a_{2n+1} = -\frac{1}{\lambda^2 - 1}\lim_{n\to\infty}(x_{2n+2} + \lambda x_{2n+1}) = a.$$

According to Theorem 2.3.9, $\lim_{n\to\infty} a_n = a$.

(2) Assume that $4a_{n+2} - 4a_{n+1} + a_n \to a$. Observe that

$$\begin{aligned} 4a_{n+2} - 4a_{n+1} + a_n &= 4\left(a_{n+2} - a_{n+1} + \frac{1}{4}a_n\right) \\ = 4\left[\left(a_{n+2} - \frac{1}{2}a_{n+1}\right) - \frac{1}{2}\left(a_{n+1} - \frac{1}{2}a_n\right)\right] &:= 4\left(y_{n+1} - \frac{1}{2}y_n\right). \end{aligned}$$

Hence

$$y_{n+1} - \frac{1}{2}y_n \to \frac{1}{4}a = \left(1 - \frac{1}{2}\right)\frac{a}{2}.$$

By (1), we must have

$$\lim_{n\to\infty} y_n = \frac{a}{2} \quad \text{or} \quad a_{n+1} - \frac{1}{2}a_n \to \left(1 - \frac{1}{2}\right)a.$$

Using (1) again yields $a_n \to a$. □

**Example 2.2.18.** Define

$$a_{n+1} := \sin a_n, \quad n \in \mathbb{N} \text{ and } 0 < a_0 < \pi.$$

Show that

$$a_n \text{ is decreasing and } \lim_{n\to\infty} a_n = 0.$$

Moreover

$$\lim_{n\to\infty} \frac{a_n}{\sqrt{3/n}} = 1.$$

PROOF. $a_1 = \sin a_0 \in (0, \pi)$. In general, we have $0 < a_n < \pi$. Since $\sin x < x$ for all $x \in (0, \pi)$, it follows that $a_{n+1} < a_n$. By Theorem 2.3.1, the limit $\lim)n \to \infty a_n$ exists, says $\alpha \in [0, \pi)$. Hence $\alpha = \sin\alpha$ which implies $\alpha = 0$.

By Theorem 2.2.9,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{na_n^2} &= \lim_{n\to\infty} \frac{\frac{1}{a_n^2}}{n} = \lim_{n\to\infty} \frac{\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}}{(n+1) - n} \\ &= \lim_{n\to\infty}\left(\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{\sin^2 a_n} - \frac{1}{a_n^2}\right) = \frac{1}{3}. \end{aligned}$$

Here we used the fact that

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) = \lim_{x\to 0}\frac{x^2 - \sin^2 x}{x^2\sin^2 x} = \lim_{x\to 0}\frac{\frac{x^4}{3}}{x^4} = \frac{1}{3}$$

and $\sin^2 x \sim (x - \frac{x^3}{6} + o(x^4))^2 \sim x^2 - \frac{x^4}{3} + \cdots$. □

### 2.3. Convergence tests

The most important test is the so-called Cauchy criterion which gives a necessary and sufficient condition on the convergence of a given sequence.

**2.3.1. Monotone sequences.** A sequence $\{a_n\}_{n\geq 1}$ is said to be **(monotonically) increasing** (resp. **decreasing**) if $a_n \leq a_{n+1}$ (resp. $a_n \geq a_{n+1}$) for all $n = 1, 2, \cdots$.

**Theorem 2.3.1.** *Suppose that* $\{a_n\}_{n\geq 1}$ *is monotonic (i.e., monotonically increasing or decreasing). Then*

$$\{a_n\}_{n\geq 1} \text{ is convergent} \iff \{a_n\}_{n\geq 1} \text{ is bounded.} \tag{2.3.1}$$

PROOF. $\Longrightarrow$: clearly.

$\Longleftarrow$: WLOG, we may assume that $a_n \leq a_{n+1}$. Let $E := \{a_n : n = 1, 2, \cdots\}$. If $\{a_n\}_{n\geq 1}$ is bounded, then by Zorn's lemma $a := \sup E$ exists and hence $a_n \leq a$.

$\forall\, \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $a - \epsilon < a_N \leq a$, for otherwise, $a - \epsilon$ would be an upper bound of $E$. Since $a_n$ is increasing, we get that $a - \epsilon < a_n \leq a$ for all $n > N$. ☐

**Example 2.3.2.** (1) $a_1 := \sqrt{2}$, $a_{n+1} := \sqrt{2 + a_n}$ $(n \geq 1) \Longrightarrow$ Find $\lim_{n\to\infty} a_n$.

(2) $a_1 > 0$, $a_{n+1} = \frac{1}{2}(a_n + \frac{1}{a_n})$ $(n \geq 1) \Longrightarrow$ Find $\lim_{n\to\infty} a_n$.

PROOF. (1) Observe that

- If $\lim_{n\to\infty} a_n = a$, then "$a = \sqrt{2 + a}$" $\Longrightarrow (a - 2)(a + 1) = 0 \Longrightarrow a = 2$.
- $a_2 = \sqrt{2 + a_1} = \sqrt{2 + \sqrt{2}} > a_1$, $a_2 < \sqrt{2 + 2} = 2$; $a_3 = \sqrt{2 + a_2} > \sqrt{2a_2} > a_2$.

In general, we claim that

$$\sqrt{2} \leq a_n < 2 \quad \text{and} \quad a_{n+1} > a_n.$$

In fact, $\sqrt{2} \leq a_n < 2 \Longrightarrow a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = 2$, and $a_{n+1} > a_n \Longrightarrow a_{n+2} = \sqrt{2 + a_{n+1}} > \sqrt{2a_{n+1}} > a_{n+1}$. Hence $\{a_n\}_{n\geq 1}$ is monotonically increasing and bounded $\Longrightarrow \lim_{n\to\infty} a_n = 2$.

(2) $\forall n \geq 1$, we have $a_n > 0$ and

$$a_{n+1} - 1 = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right) - 1 = \frac{1}{2}\left(\sqrt{a_n} - \frac{1}{\sqrt{a_n}}\right)^2 \geq 0.$$

On the other hand,

$$a_{n+1} \leq \frac{1}{2}(a_n + a_n) = a_n.$$

Hence $a_n \geq a_{n+1} \geq \cdots \geq 1 \Longrightarrow \lim_{n\to\infty} a_n = a$ exists and $a \geq 1$. Solving the equation $a = \frac{1}{2}(a + \frac{1}{a})$ yields $a = 1$. ☐

**Example 2.3.3.** $a_1 = 1$, $a_{n+1} = \frac{1}{1+a_n}$ $(n \geq 1) \Longrightarrow$ Find $\lim_{n\to\infty} a_n$.

PROOF. Observe

$$a_2 = \frac{1}{2}, \quad a_3 = \frac{2}{3}, \quad a_4 = \frac{3}{5}, \quad a_5 = \frac{5}{8}, \quad \cdots.$$

We claim that $\{a_{2n}\}_{n\geq 1}$ is increasing but $\{s_{2n-1}\}_{n\geq 1}$ is decreasing, and $\frac{1}{2} \leq a_n \leq 1$. Indeed, $\frac{1}{2} \leq a_n \leq 1 \Longrightarrow a_{n+1} \geq \frac{1}{1+1} = \frac{1}{2}$ and $a_{n+1} \leq \frac{1}{1+0} = 1$. Moreover

$$a_{2n+2} = \frac{1}{1+a_{2n+1}} \geq \frac{1}{1+a_{2n-1}} = a_{2n}, \quad a_{2n+1} = \frac{1}{1+a_{2n}} \leq \frac{1}{1+a_{2n-2}} = a_{2n-1}.$$

Let $\lim_{n\to\infty} a_{2n} = A$ and $\lim_{n\to\infty} a_{2n-1} = B \Longrightarrow B = 1/(1+A)$ and $A = 1/(1+B)$ $\Longrightarrow A = B = (\sqrt{5}-1)/2$. Thus $a_n \to (\sqrt{5}-1)/2$. □

**2.3.2. Three important constants $\pi$, $e$, and $\gamma$.** Recall that $\pi = 3.1415926\cdots$ and $e = 2.7182818284590\cdots$.

**A. Constant $\pi$.** The following theorem is will-known.

**Theorem 2.3.4. (Euler, 1734)** *We have*

$$\sum_{n\geq 1} \frac{1}{n^2} := \lim_{N\to\infty} \sum_{1\leq n\leq N} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{2.3.2}$$

PROOF. (1) The first proof is due to John Scholes.

**Claim 1:** For any $m \in \mathbb{N}$,

$$\cot^2\left(\frac{\pi}{2m+1}\right) + \cdots + \cot^2\left(\frac{m\pi}{2m+1}\right) = \frac{2m(2m-1)}{6}.$$

Consider

$$\cos(nx) + \mathbf{i}\sin(nx) = e^{\mathbf{i}nx} = (e^{\mathbf{i}x})^n = (\cos x + \mathbf{i}\sin x)^n.$$

The imaginary part yields

$$\sin(nx) = \binom{n}{1}\sin x\cos^{n-1}x - \binom{n}{3}\sin^3 x\cos^{n-3}x \pm \cdots.$$

Let $n := 2m+1$ and $x = \frac{r\pi}{2m+1}$ $(1 \leq r \leq m) \Longrightarrow$

$$0 = \sin(nx) = \binom{n}{1}\sin x\cos^{n-1}x - \binom{n}{3}\sin^3 x\cos^{n-3}x \pm \cdots.$$

Divided by $\sin^n x$ $(0 < x < \frac{\pi}{2})$ we get

$$\begin{aligned} 0 &= \binom{n}{1}\cot^{n-1}x - \binom{n}{3}\cot^{n-3}x \pm \cdots \\ &= \binom{2m+1}{1}\cot^{2m}x - \binom{2m+1}{3}\cot^{2m-2}x \pm \cdots. \end{aligned}$$

Let

$$P(t) := \binom{2m+1}{1}t^m - \binom{2m+1}{3}t^{m-1} \pm \cdots + (-1)^m\binom{2m+1}{2m+1}.$$

This polynomial has $m$ different roots

$$a_r := \cot^2\left(\frac{r\pi}{2m+1}\right), \quad 1 \le r \le m.$$

Therefore

$$P(t) = \binom{2m+1}{1}\prod_{1\le r\le m}\left[t - \cot^2\left(\frac{r\pi}{2m+1}\right)\right].$$

In particular

$$\sum_{1\le r\le m} a_r = \frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{2m(2m-1)}{6}.$$

**Claim 2:** One has

$$\sum_{1\le r\le m}\csc^2\left(\frac{r\pi}{2m+1}\right) = \frac{2m(2m+2)}{6}.$$

Indeed,

$$\sum_{1\le r\le m}\csc^2\left(\frac{r\pi}{2m+1}\right) = \sum_{1\le r\le m}\frac{1}{\sin^2(\frac{r\pi}{2m+1})}$$

$$= \sum_{1\le r\le m}\left[1 + \cot^2\left(\frac{r\pi}{2m+1}\right)\right] = m + \frac{2m(2m-1)}{6}.$$

In the interval $(0, \pi/2)$, the following relations hold:

$$0 < \sin y < y < \tan y, \quad 0 < \cot y < \frac{1}{y} < \csc y, \quad 0 < \cot^2 y < \frac{1}{y^2} < \csc^2 y.$$

Consequently,

$$\frac{2m(2m-1)}{6} < \sum_{1\le r\le m}\left(\frac{2m+1}{r\pi}\right)^2 < \frac{2m(2m+2)}{6}.$$

Equivalently

$$\frac{\pi^2}{6}\frac{2m}{2m+1}\frac{2m-1}{2m+1} < \sum_{1\le r\le m}\frac{1}{r^2} < \frac{\pi^2}{6}\frac{2m}{2m+1}\frac{2m+2}{2m+1}.$$

Finally letting $m \to \infty$ yields $\sum_{n\ge1}\frac{1}{n^2} = \pi^2/6$.

(2) The second proof is due to Beukers-Calabi-Kolk. Observe

$$\sum_{n\ge1}\frac{1}{n^2} = \sum_{n\ge0}\frac{1}{(2n-1)^2} + \sum_{n\ge1}\frac{1}{(2n)^2}.$$

Hence

$$\sum_{n\ge1}\frac{1}{n^2} = \frac{\pi^2}{6} \iff \sum_{k\ge0}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Define

$$J := \iint_{[0,1]\times[0,1]}\frac{dxdy}{1-x^2-y^2} = \sum_{k\ge0}\frac{1}{(2k+1)^2}.$$

If

$$u = \cos^{-1}\sqrt{\frac{1-x^2}{1-x^2y^2}}, \quad v := \cos^{-1}\sqrt{\frac{1-y^2}{1-x^2y^2}},$$

then $x = \sin u/\cos v$, $y = \sin v/\cos u$, and

$$J = \int_0^{\pi/2}\int_0^{\pi/2-u} dudv = \frac{\pi^2}{8}.$$

□

**B. Constant $e$.** Define

$$a_n := \left(1+\frac{1}{n}\right)^n, \quad b_n := \left(1+\frac{1}{n}\right)^{n+1}, \quad e_n := 1 + \sum_{1\le k\le n}\frac{1}{k!} = \sum_{0\le k\le n}\frac{1}{k!}.$$

**Claim 1:** For each $n$,

$$a_n < a_{n+1}, \quad b_n > b_{n+1}.$$

PROOF. For each $n$,

$$\begin{aligned} a_n &= \left(1+\frac{1}{n}\right)^n = \sum_{0\le k\le n}\binom{n}{k}\frac{1}{n^k} = 1 + \sum_{1\le k\le n}\frac{n(n-1)\cdots(n-k)}{k!}\frac{1}{n^k} \\ &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \\ &< 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right) \\ &\quad + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right) = a_{n+1}. \end{aligned}$$

For $b_n$,

$$\frac{b_{n-1}}{b_n} = \frac{(1+\frac{1}{n-1})^n}{(1+\frac{1}{n})^{n+1}} = \left(\frac{1+\frac{1}{n-1}}{1+\frac{1}{n}}\right)^n \frac{1}{1+\frac{1}{n}}$$

$$= \left(1+\frac{1}{n^2-1}\right)^n\frac{1}{1+\frac{1}{n}} > \left(1+\frac{n}{n^2-1}\right)\frac{1}{1+\frac{1}{n}} > \left(1+\frac{1}{n}\right)\frac{1}{1+\frac{1}{n}} = 1.$$

□

**Claim 2:** We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n := e \tag{2.3.3}$$

exists.

PROOF. Because

$$a_n < 1+1+\sum_{2\le k\le n}\frac{1}{k!} \le 2 + \sum_{2\le k\le n}\frac{1}{k(k-1)} = 3 - \frac{1}{n} < 3,$$

the claim follows from Theorem 2.3.1 and **Claim 1**. □

**Claim 3:** For any $n \in \mathbb{Z}_+$,

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} < e < \left(1+\frac{1}{n+1}\right)^{n+2} < \left(1+\frac{1}{n}\right)^{n+1}. \tag{2.3.4}$$

**Claim 4:** We have

$$\lim_{n\to\infty} e_n = e. \tag{2.3.5}$$

PROOF. Observe that $e_n < e_{n+1}$ and

$$\begin{aligned} a_n &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \\ &> 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right). \end{aligned}$$

Letting $n \to \infty$ yields

$$e \geq 1+1+\frac{1}{2!}+\cdots+\frac{1}{k!} = e_k.$$

On the other hand, $a_n < e_n$. So $\lim_{n\to\infty} e_n = e$. □

**Example 2.3.5.** (1) $\forall n \geq 1 \Longrightarrow$

$$\left(\frac{n+1}{e}\right)^n < n! < e\left(\frac{n+1}{e}\right)^{n+1}. \tag{2.3.6}$$

(2) We have

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}. \tag{2.3.7}$$

PROOF. (1) $\forall k \geq 1$,

$$\left(\frac{k+1}{k}\right)^k < e < \left(\frac{k+1}{k}\right)^{k+1}.$$

So

$$\frac{(n+1)^n}{n!} = \prod_{1\leq k\leq n}\left(\frac{k+1}{k}\right)^k < e^n < \prod_{1\leq k\leq n}\left(\frac{k+1}{k}\right)^{k+1} = \frac{(n+1)^{n+1}}{n!}.$$

(2) By (1), one has

$$\frac{n+1}{e} < \sqrt[n]{n!} < \frac{n+1}{e}\sqrt[n]{n+1}$$

and

$$\frac{n+1}{n}\cdot\frac{1}{e} < \frac{\sqrt[n]{n!}}{n} < \frac{n+1}{n}\cdot\sqrt[n]{n+1}\cdot\frac{1}{e}.$$

Letting $n \to \infty$ yields (2.3.7). □

**C. Euler constant $\gamma$.** Given $p > 0$ and let

$$S_n := \sum_{1 \le k \le n} \frac{1}{k^p}, \quad n \in \mathbb{Z}_+.$$

Then $S_n < S_{n+1}$, and

$$\begin{aligned} S_n &\le S_{2^n - 1} \\ &= 1 + \underbrace{\left(\frac{1}{2^p} + \frac{1}{3^p}\right)}_{<2^{-(p-1)}} + \underbrace{\left(\frac{1}{4^p} + \cdots + \frac{1}{7^p}\right)}_{<4^{-(p-1)} = 2^{-2(p-1)}} + \cdots + \underbrace{\left(\frac{1}{2^{(n-1)p}} + \cdots + \frac{1}{(2^n - 1)^p}\right)}_{<2^{-(n-1)(p-1)}} \\ &< \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{2^{p-1}}{2^{p-1} - 1}. \end{aligned}$$

Hence

$$\lim_{n \to \infty} S_n \text{ exists for all } p > 1.$$

When $p = 1$, by Theorem 2.2.9,

$$\begin{aligned} \lim_{n \to \infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{\ln n} &= \lim_{n \to \infty} \frac{\frac{1}{n}}{\ln n - \ln(n-1)} = \lim_{n \to \infty} \frac{\frac{1}{n}}{\ln(1 + \frac{1}{n-1})} \\ &= \lim_{n \to \infty} \frac{\frac{1}{n}}{\ln(1 + \frac{1}{n})} \cdot \lim_{n \to \infty} \frac{\ln(1 + \frac{1}{n})}{\ln(1 + \frac{1}{n-1})} = 1, \end{aligned}$$

because

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}.$$

Consequently

$$S_n \ge 1 + \frac{1}{2} + \cdots + \frac{1}{n} \to +\infty, \quad \text{if } 0 < p \le 1.$$

In particular

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \sim \ln n \quad \text{as } n \to \infty.$$

Define

$$a_n := \sum_{1 \le k \le n} \frac{1}{k} - \ln n. \tag{2.3.8}$$

Then

$$a_n > a_{n+1} > 0, \quad \lim_{n \to \infty} a_n \text{ exists.}$$

Indeed,

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n > \ln \frac{2}{1} + \ln \frac{3}{2} + \cdots + \ln \frac{n+1}{n} - \ln n = \ln \frac{n+1}{n} > 0$$

and

$$a_{n+1} - a_n = \frac{1}{n+1} - \ln(n+1) + \ln n = \frac{1}{n+1} - \ln\left(1 + \frac{1}{n}\right) < 0.$$

Figure: Joseph Liouville (1809/3/24 - 1882/9/8)

**Definition 2.3.6.** The **Euler constant** $\gamma$ is defined to be

$$\gamma := \lim_{n\to\infty}\left(\sum_{1\le k\le n}\frac{1}{k} - \ln n\right). \tag{2.3.9}$$

**Conjecture 2.3.7.** *$\gamma$ is irrational, i.e., $\gamma \in \mathbb{R}\setminus\mathbb{Q}$.*

**Theorem 2.3.8.** *(1)* **(Liouville, 1840)** *$e$ is irrational.*
*(2) $\pi$ is irrational.*

PROOF. (1) Recall that

$$e = \sum_{k\ge 0}\frac{1}{k!} = \lim_{n\to\infty}\sum_{0\le k\le n}\frac{1}{k!}.$$

Assume that $e = a/b$ is rational, with $a, b > 0$. Then

$$n!be = n!a, \quad \forall\, n \in \mathbb{N}.$$

On the other hand,

$$\begin{aligned} bn!e &= bn!\left[\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)+\left(\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots\right)\right] \\ &= bn!\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right) \\ &\quad + b\underbrace{\left(\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\cdots\right)}_{\frac{1}{n+1}<?<\frac{1}{n+1}+\frac{1}{(n+1)^2}+\cdots=\frac{1}{n}} \end{aligned}$$

When $n$ is large enough, the second term is not an integer. This contradiction shows that $e$ is not rational.

(2) We give a proof due to Niven (1946). Let $\pi = a/b$. Define

$$f(x) := \frac{x^n(a-bx)^n}{n!}, \quad F(x) := f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x).$$

But

$$\mathbb{Z} \ni F(\pi) + F(0) = \int_0^\pi f(x)\sin x\, dx$$

with $0 < f(x)\sin x < \frac{\pi^n a^n}{n!} \ll 1$ (as $n \gg 1$). □

**2.3.3. Subsequences.** Let $\{a_n\}_{n\geq 1}$ be a sequence and $\varphi : \mathbb{N} \to \mathbb{N}$ s strictly increasing function. Then $\{a_{\varphi(k)}\}_{k\geq 1}$ is called a **subsequence** and write $\{a_{n_k}\}_{k\geq 1}$.

**Theorem 2.3.9.** *(1) If $\lim_{n\to\infty} a_n = a$, then for any subsequence $\{a_{n_k}\}_{k\geq 1}$ one has*

$$\lim_{k\to\infty} a_{n_k} = a.$$

*(2) $\{a_n\}_{n\geq 1}$ is convergent $\Longrightarrow$ each subsequence is convergent.*
*(3) $\exists$ divergent subsequence of $\{a_n\}_{n\geq 1} \Longrightarrow \{a_n\}_{n\geq 1}$ is divergent.*
*(4) $\exists$ two convergent subsequences with distinct limits $\Longrightarrow \{a_n\}_{n\geq 1}$ is divergent.*
*(5) $\{a_n\}_{n\geq 1}$ is convergent $\Longleftrightarrow \{a_{2n-1}\}_{n\geq 1}$ and $\{a_{2n}\}_{n\geq 1}$ are convergent and have the same limits.*

PROOF. (1) - (4) can be proved by directly definition. For (5), assume that $\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} a_{2n} = a$. Then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$|b_n - a| < \epsilon, \quad |c_n - a| < \epsilon, \quad b_n := a_{2n},\ c_n := a_{2n-1}.$$

For $a_n$, if $n = 2k$, then $|a_n - a| < \epsilon$ $(n > 2N)$; if $n = 2k-1$, then $|a_n - a| < \epsilon$ $(n > 2N-1)$. □

**Example 2.3.10. (Fibonacci sequence)** Let

$$a_1 = a_2 = 1, \quad a_{n+1} = a_n + a_{n-1}\ (n \geq 2) \quad \Longrightarrow \quad \text{find } \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Let

$$b_n := \frac{a_{n+1}}{a_n}.$$

Then

$$b_n = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n} = 1 + \frac{1}{b_{n-1}}.$$

We have proved as in Example 2.3.3 that

$$b_{2n-1} < b_{2n+1}, \quad b_{2n} > b_{2n+2}, \quad 1 \leq b_n \leq 2.$$

Then

$$\lim_{n\to\infty} b_n = \frac{\sqrt{5}+1}{2}, \quad \lim_{n\to\infty}(b_n - 1) = \lim_{n\to\infty} \frac{1}{b_{n-1}} = \frac{\sqrt{5}-1}{2} \approx 0.618.$$

The explicit expression for $a_n$ is

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

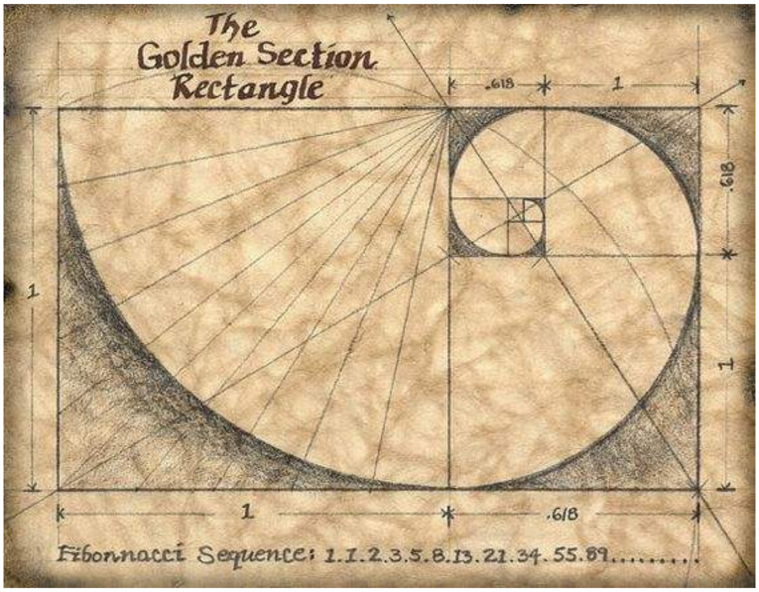



Figure: Fibonacci sequence

Suppose

$$a_n - \alpha a_{n-1} = \beta(a_{n-1} - \alpha a_{n-2}).$$

Then

$$\alpha + \beta = 1, \quad \alpha\beta = -1$$

so that $(\alpha, \beta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ or $((1-\sqrt{5})/2, (1+\sqrt{5})/2)$. From

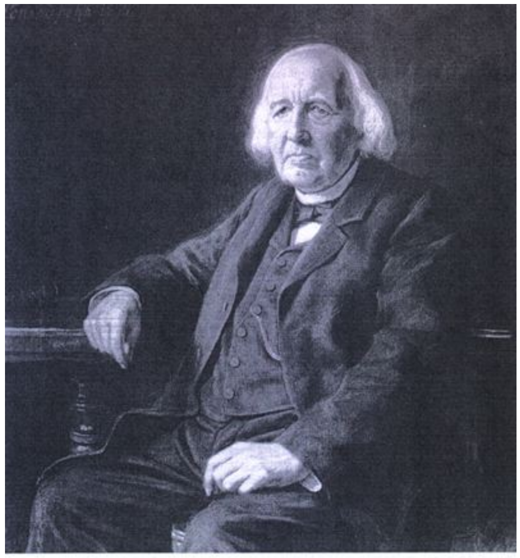
$$\begin{aligned} a_n - \frac{1+\sqrt{5}}{2}a_{n-1} &= \frac{1-\sqrt{5}}{2}\left(a_{n-1} - \frac{1+\sqrt{5}}{2}a_{n-2}\right), \\ a_n - \frac{1-\sqrt{5}}{2}a_{n-1} &= \frac{1+\sqrt{5}}{2}\left(a_{n-1} - \frac{1-\sqrt{5}}{2}a_{n-2}\right) \end{aligned}$$

we obtain

$$\begin{aligned} a_n - \frac{1-\sqrt{5}}{2}a_{n-1} &= \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(a_2 - \frac{1-\sqrt{5}}{2}a_1\right), \\ a_n - \frac{1+\sqrt{5}}{2}a_{n-1} &= \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(a_2 - \frac{1+\sqrt{5}}{2}a_1\right). \end{aligned}$$

Eliminating $a_{n-1}$ yields the formula for $a_n$.

**Theorem 2.3.11. (Bolzano-Weierstrass theorem)** *Every bounded sequence has a convergent subsequence.*



Karl Weierstraß 1815–1897

Figure: Karl Theodor Wihelm Weierstrass (1815/10/31 - 1897/2/19)

PROOF. Assume $\{a_n\}_{n\geq 1}$ is bounded, i.e., $a_n \in [a, b]$ for some interval $[a, b]$ and all $n \geq 1$. Then one of $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ contains infinitely many $a_n$'s, say $[a_1, b_1]$. Take $x_{n_1} \in [a_1, b_1]$. In this process, we can find a sequence of closed intervals

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_k, b_k] \supset \cdots$$

with

$$b_k - a_k = \frac{b-a}{2^k} \to 0 \quad \text{and} \quad \exists x_{n_k} \in [a_k, b_k].$$

But $a_n$ is increasing and $b_n$ is decreasing, we conclude that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ both exist. Moreover

$$0 \leq b - a \leq b_n - a_n \to 0.$$

Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$$

and $c \in [a_k, b_k]$ for each $k$. From $|x_{n_k} - 1| \leq b_k - a_k$, we see that $\lim_{n\to\infty} x_{n_k} = c$. □

**Theorem 2.3.12.** *$\{a_n\}_{n\geq 1}$ is unbounded $\Longrightarrow$ $\exists$ subsequence $\{a_{n_k}\}_{k\geq 1}$ such that $\{a_{n_k}\}_{k\geq 1}$ is unbounded.*

PROOF. $\exists$ $n_1$ such that $|a_{n_1}| > 1$. Then $\exists n_2 > n_1$ such that $|a_{n_2}| > 2$. Hence $\exists$ subsequence $\{n_k\}_{k\geq 1}$ such thst $|a_{n_k}| \geq k$. □

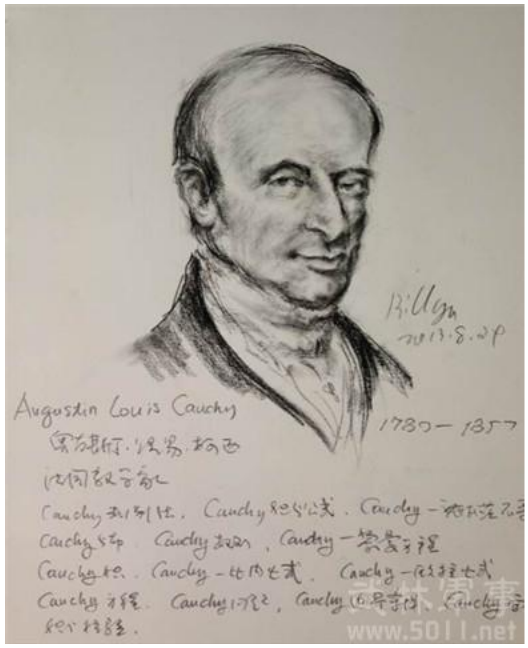



Figure: Augustin Louis Cauchy (1789/8/21 - 1857/5/23)

**2.3.4. Cauchy sequences.** We say that a sequence $\{a_n\}_{n\geq 1}$ is a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ whenever $n, m \geq N$.

**Example 2.3.13.** (1) $\{a_n\}_{n\geq 1}$ is not Cauchy, where $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

(2) $\{a_n\}_{n\geq 1}$ is Cauchy, where $a_n = 1 + \frac{1}{2\sqrt{2}} + \cdots + \frac{1}{n\sqrt{n}}$.

PROOF. (1) $\forall\, n \geq 1$,

$$a_{2n} - a_n = \frac{1}{n+1} + \cdots + \frac{1}{2n} \geq \frac{n}{2n} = \frac{1}{2}.$$

(2) $\forall n \geq 1$,

$$\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}} = \frac{1}{\sqrt{n(n+1)}} \cdot \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{2}\frac{1}{(n+1)\sqrt{n+1}}.$$

Hence $\forall\, m > n$,

$$a_m - a_n = \frac{1}{(n+1)\sqrt{n+1}} + \cdots + \frac{1}{m\sqrt{m}} < \frac{2}{\sqrt{n}} - \frac{2}{\sqrt{m}} < \frac{2}{\sqrt{n}}.$$

□

**Remark 2.3.14.** Cauchy sequence is bounded.

PROOF. $\exists\, N \in \mathbb{N}$ such that $|a_m - a_n| < 1$ whenever $m, n \geq N$. □

**Theorem 2.3.15. (Cauchy criterion)** $\{a_n\}_{n\geq 1}$ *is convergent* $\Longleftrightarrow$ $\{a_n\}_{n\geq 1}$ *is Cauchy.*

PROOF. $\Longrightarrow$: Let $\lim_{n\to\infty} a_n = a$. Then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon, \quad \forall\, n > N.$$

Hence

$$|a_m - a_n| \leq |a_m - a| + |a_n - a| < 2\epsilon$$

for all $n, m > N$.

$\Longleftarrow$: $\exists N_0 \in \mathbb{N}$ such that $|a_n - a_{N_0+1}| < 1, \forall n > N_0$. In particular, $|a_n| \leq M$. By Theorem 2.3.11, $\exists$ subsequence $\{a_{n_k}\}_{k\geq 1}$ such that $\lim_{k\to\infty} a_{n_k} = a$. Furthermore

$$|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \epsilon + |a_{n_k} - a|$$

whenever $n, n_k > N$. □

**Remark 2.3.16.** The above theorem may not be true for any metric spaces. For example,

$$x_0 = 2, \quad x_{n+1} := \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \in \mathbb{Q}.$$

Then $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{Q}$, but $x_n \to \sqrt{2} \notin \mathbb{Q}$.

**Remark 2.3.17.** We have defined that a metric space is complete if $\forall$ Cauchy sequence is convergent. Hence $\mathbb{R}$ is complete, but $\mathbb{Q}$ is not.

**Remark 2.3.18.** "Completeness" can be specialized on Riemannian manifolds so that the completeness is, in this case, equivalent to the fact that every geodesic can be extended for any time.

CHAPTER 3

# Continuous functions

## 3.1. Limits of functions

We begin with an example. Find a function $f : \mathbb{R} \to \mathbb{R}_+$ satisfying

$$f(x+y) = f(x)f(y)$$

for any $x, y \in \mathbb{R}$? Letting $x = y$ yields

$$f(2) = f(1+1) = [f(1)]^2.$$

In general, one has

$$f(n) = [f(1)]^n$$

for any $n \in \mathbb{Z}_+$. Moreover

$$f(1) = f\Bigg(\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{n}\Bigg) = \left[f\left(\frac{1}{n}\right)\right]^n \quad \text{and} \quad f(1) = f(1)f(0)$$

so that

$$f(0) = 1, \quad f\left(\frac{1}{n}\right) = [f(1)]^{\frac{1}{n}}.$$

For any $p/q \in \mathbb{Q}$ one has

$$f\left(\frac{p}{q}\right) = f\Bigg(\underbrace{\frac{1}{q} + \cdots + \frac{1}{q}}_{p}\Bigg) = \left[f\left(\frac{1}{q}\right)\right]^p = [f(1)]^{\frac{p}{q}}.$$

For any $x \in \mathbb{R}$, we have proved that there is a sequence $\{a_n\}_{n\geq 1}$ in $\mathbb{Q}$ such that

$$\lim_{n\to\infty} a_n = x.$$

In summary,

$$f(x) \xleftarrow{\text{we expect}} f(a_n) = [f(1)]^{a_n} \xrightarrow{n\to\infty} [f(1)]^x$$

If one can prove

$$f\left(\lim_{n\to\infty} a_n\right) = f(x) = \lim_{n\to\infty} f(a_n),$$

we have

$$f(x) = [f(1)]^x, \quad x \in \mathbb{R}.$$

We can conclude that when $f$ and lim can be changed, $f(x) = [f(1)]^x$ for all $x \in \mathbb{R}$. As we shall see later by Heine's theorem, $f(x) = [f(1)]^x$ for all $x \in \mathbb{R}$ when $f$ is continuous.

**3.1.1. Definitions.** We start with the definition of limits of functions.

**Definition 3.1.1.** (1) If $f : (a, +\infty) \to \mathbb{R}$ and $A \in \mathbb{R}$, we define

$$\lim_{x\to+\infty} f(x) = A \iff \begin{array}{c} \forall\, \epsilon > 0\ \exists\, M > a \text{ such that} \\ |f(x) - A| < \epsilon \\ \text{whenever } x > M \end{array} \tag{3.1.1}$$

(2) If $f : (-\infty, b) \to \mathbb{R}$ and $A \in \mathbb{R}$, we define

$$\lim_{x\to-\infty} f(x) = A \iff \begin{array}{c} \forall\, \epsilon > 0\ \exists\, M < b \text{ such that} \\ |f(x) - A| < \epsilon \\ \text{whenever } x < M \end{array} \tag{3.1.2}$$

(3) If $f : \mathbb{R} \to \mathbb{R}$ and $A \in \mathbb{R}$, we define

$$\lim_{x\to\infty} f(x) = A \iff \begin{array}{c} \forall\, \epsilon > 0\ \exists\, M > 0 \text{ such that} \\ |f(x) - A| < \epsilon \\ \text{whenever } |x| > M \end{array} \tag{3.1.3}$$

**Theorem 3.1.2.** *If $f : \mathbb{R} \to \mathbb{R}$ and $A \in \mathbb{R}$, then*

$$\lim_{x\to\infty} f(x) = A \iff \lim_{x\to+\infty} f(x) = A = \lim_{x\to-\infty} f(x). \tag{3.1.4}$$

**Example 3.1.3.** Compute

$$\lim_{x\to\infty} \frac{e^x}{1+e^x}, \quad \lim_{x\to\infty} \frac{\sin x}{x}.$$

Since

$$\lim_{x\to+\infty} \frac{e^x}{1+e^x} = 1 \quad \text{and} \quad \lim_{x\to-\infty} \frac{e^x}{1+e^x} = 0,$$

it follows that the limit $\lim_{x\to\infty} \frac{e^x}{1+e^x}$ does not exist. For the second one,

$$\left|\frac{\sin x}{x}\right| \le \frac{1}{|x|},$$

we see that $\lim_{x\to\infty} \sin x/x = 0$.

**Definition 3.1.4.** Consider a deleted neighborhood $\mathring{U}(a, \rho) := (a - \rho, a) \cup (a, a + \rho)$. Let $f : \mathring{U}(a, \rho) \to \mathbb{R}$, $A \in \mathbb{R}$, define

$$\lim_{x\to a} f(x) = A \iff \begin{array}{c} \forall\, \epsilon > 0\ \exists\, \delta > 0\ (\delta \le \rho) \text{ such that} \\ |f(x) - A| < \epsilon \\ \text{whenever } 0 < |x - a| < \delta. \end{array} \tag{3.1.5}$$

If $f$ is defined at $a$, then $\lim_{x\to a} f(x) = f(a)$ is the definition of the **continuity of $f$ at $a$**.

**Definition 3.1.5.** **(One-sided limits)** (1) $f : (a-\rho, a) \to \mathbb{R}$ ($\rho > 0$), $A \in \mathbb{R}$, define

$$\lim_{x\to a-} f(x) \equiv f(a-) = A \iff \begin{array}{c} \forall\, \epsilon > 0\ \exists\, \delta > 0 \text{ such that} \\ |f(x) - A| < \epsilon \\ \text{whenever } -\delta < x - a < 0 \end{array} \tag{3.1.6}$$

(2) $f : (a, a+\rho) \to \mathbb{R}$ ($\rho > 0$), $A \in \mathbb{R}$, define

$$\lim_{x\to a+} f(x) \equiv f(a+) = A \iff \begin{array}{c} \forall\, \epsilon > 0\ \exists\, \delta > 0 \text{ such that} \\ |f(x) - A| < \epsilon \\ \text{whenever } 0 < x - a < \delta \end{array} \tag{3.1.7}$$

**Theorem 3.1.6.** *We have*

$$\lim_{x\to a} f(x) = A \quad \iff \quad f(a+) = A = f(a-). \tag{3.1.8}$$

**Example 3.1.7.** (1) If

$$f(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0, \end{cases}$$

then $f(0+) = -1$ and $f(0+) = 1$.

(2) If

$$f(x) = \begin{cases} 0, & x \le 0, \\ \frac{1}{x}, & x > 0, \end{cases}$$

then $f(0-) = 0$, but $f(0+)$ does not exist.

**3.1.2. Properties of the limits of a function.** The following properties can be proved as for the limits of sequences.

**Theorem 3.1.8.** *(1)* **(Uniqueness)** $\lim_{x\to a} f(x)$ *is unique, if it exists.*

*(2)* **(Local boundedness)** $\lim_{x\to a} f(x)$ *exists* $\Longrightarrow$ $f$ *is bounded in some delated neighborhood of* $a$.

*(3)* $\lim_{x\to a} f(x) = A > B = \lim_{x\to a} g(x) \Longrightarrow f(x) > g(x)$ *in some* $\mathring{U}(a,\rho)$.

*(4)* $\lim_{x\to a} f(x) = A \Longrightarrow \lim_{x\to a} |f(x)| = |A|$.

*(5)* $g(x) \le f(x) \le h(x)$ *for any* $x \in \mathring{U}(a,\rho) \Longrightarrow$ *if* $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = A$, *then* $\lim_{x\to a} f(x) = A$.

*(6) If* $\lim_{x\to a} f(x) = A$, $\lim_{x\to a} g(x) = B$, $\alpha, \beta \in \mathbb{R}$, *then*

$$\lim_{x\to a}(\alpha f(x) + \beta g(x)) = \alpha A + \beta B, \ \ \lim_{x\to a} f(x)g(x) = AB, \ \ \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{A}{B} \ (\text{if } B \neq 0).$$

*(7)* $\lim_{x\to x_0} g(x) = u_0$, $\lim_{u\to u_0} f(u) = A$, $g \neq u_0$ *in* $\mathring{U}(x_0,\rho) \Longrightarrow$

$$\lim_{x\to x_0} f(g(x)) = A.$$

**Example 3.1.9.** (1) $\lim_{x\to 0} x\lfloor 1/x\rfloor = 1$. Indeed, $1/x - 1 < \lfloor 1/x\rfloor \le 1/x$.

(2) $\lim_{x\to\infty} x^k/a^x = 0$ ($a > 0$ and $k \in \mathbb{N}$). Because $0 < x^k/a^x \le (\lfloor x\rfloor + 1)^k/a^{\lfloor x\rfloor+1}a$.

**3.1.3. Two important limits.** When $0 < x < \pi$, we know that $0 < \sin x < x$. The following property shows that when $x$ is very small, we can use $x$ to substitute $\sin x$.

**Proposition 3.1.10.** *One has*

$$\lim_{x\to 0}\frac{\sin x}{x} = 1. \tag{3.1.9}$$

PROOF. When $0 < x < \pi/2$, we have

$$\sin x < x < \tan x$$

so that

$$\cos x < \frac{\sin x}{x} < 1.$$

Hence $\lim_{x\to 0+}\sin x/x = 1$. Similarly, we can prove that $\lim_{x\to 0-}\sin x/x = 1$. □

**Remark 3.1.11.** (1) We have proved that

$$0 \xleftarrow{x\to\infty} \frac{\sin x}{x} \xrightarrow{x\to 0} 1$$

What about the value of the integral

$$\int_0^\infty \frac{\sin x}{x}dx \quad \left(=\frac{\pi}{2}\right)?$$

**Proposition 3.1.12.** *One has*

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e. \tag{3.1.10}$$

PROOF. For any $x \ge 1$ we have

$$\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1}$$

Using $e = \lim_{n\to\infty}(1+1/n)^n$, we get $\lim_{x\to+\infty}(1+1/x)^x = e$. Similarly, when $x \to -\infty$, we set $y := -x \to +\infty$. Then

$$\left(1+\frac{1}{x}\right)^x = \left(1-\frac{1}{y}\right)^{-y} = \left(1+\frac{1}{y-1}\right)^y = \left(1+\frac{1}{y-1}\right)^{y-1}\frac{y}{y-1} \to e$$

as $x \to -\infty$. □

**Remark 3.1.13.** (1) $\lim_{y\to 0}(1+y)^{1/y} = e$.
(2) $\lim_{y\to\infty}(1-1/y)^y = 1/e$.
(3) $\lim_{n\to\infty}[n\sin(2\pi n!e)] = 2\pi$.

PROOF. We give a proof of (3). From $e = \sum_{k\geq 0}\frac{1}{k!}$ we have

$$n!e = \underbrace{n!\sum_{0\leq k\leq n}\frac{1}{k!}}_{\in\mathbb{Z}} + \underbrace{n!\sum_{k\geq n+1}\frac{1}{k!}}_{:=\epsilon_n\to 0}.$$

Then

$$n\sin(2\pi n!e) = n\sin(2\pi\epsilon_n) = \frac{\sin(2\pi\epsilon_n)}{2\pi\epsilon_n}\frac{2\pi\epsilon_n}{1/n} \to 1\times 2\pi = 2\pi.$$

Here

$$\frac{1}{n+1} < \epsilon_n := \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots = \frac{1}{n}.$$

□

**3.1.4. Heine's theorem.** This theorem builds a bridge between limits of functions and limits of sequences.

**Theorem 3.1.14. (Heine)** $f : \mathring{U}(a,\rho) \to \mathbb{R}$, $A \in \mathbb{R} \Longrightarrow$

$$\lim_{x\to a} f(x) = A \iff \begin{array}{c}\forall\ \{a_n\}_{n\geq 1} \subset \mathring{U}(a,\rho) \text{ with } a_n \to a \\ \text{we have } \lim_{n\to\infty} f(a_n) = A.\end{array} \tag{3.1.11}$$

$$\iff \begin{array}{c}\forall\ \{a_n\}_{n\geq 1} \subset \mathring{U}(a,\rho) \text{ with } a_n \to a \\ \text{we have } \{f(a_n)\}_{n\geq 1} \text{ converges.}\end{array} \tag{3.1.12}$$

PROOF. (1) $\Leftarrow$: If $\lim_{x\to a} f(x) \neq A$, then $\exists\epsilon_0 > 0$, $\forall\delta > 0$, $\exists x \in \mathring{U}(a,\delta)$ such that

$$|f(x) - A| \geq \epsilon_0 > 0.$$

Take $\delta_1 = \rho$, $\delta_2 = \rho/2$, $\cdots$, $\delta_n = \rho/n$, $\cdots$, and find $a_1, \cdots, a_n \in \mathring{U}(a,\rho/n)$ such that $|f(a_n) - A| \geq \epsilon_0$. Since $a_n \to a$, we have $f(a_n) \nrightarrow A$.

$\Rightarrow$: Clearly.

(2) $\Rightarrow$: Clearly.

$\Leftarrow$: We should prove that any sequence $(f(a_n))_{n\geq 1}$ has the same limit. Suppose that $a_n \to a$ and $b_n \to a$, but $f(a_n) \to A \neq B \leftarrow f(b_n)$. Consider the new sequence $(x_n)_{n\geq 1}$:

$$a_1, b_1, a_2, b_2, a_3, b_3, \cdots, a_n, b_n, \cdots.$$

Then $x_n \to a$ but $f(x_n)_{n\geq 1}$ diverges. Hence $A$ must be equal to $B$. □

**Example 3.1.15.** (1) $\sin \frac{1}{x}$ has no limit at $x = 0$.

(2) Dirichlet function has no limit at every point $x \in \mathbb{R}$.

PROOF. (1) For $x_n = 1/n\pi$ and$y_n = 1/(2n\pi + \frac{\pi}{2})$, we get

$$\sin \frac{1}{x_n} = 0, \quad \sin \frac{1}{y_n} = 1.$$

Hence $\sin \frac{1}{x}$ has no limit at $x = 0$.

(2) Recall that

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

For any $a \in \mathbb{R}$, we can find a sequence $\{a_n\}_{n\geq 1}$ such that $a_n \to a$. On the other hand, $\exists b_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $b_n \to a$. But $D(a_n) = 1 \neq 0 = D(b_n)$. □

(3) Topologist's sine curve:

$$X = \left\{ \left( x, \sin \frac{1}{x} \right) : x \in (0, 1] \right\}$$

which can be obtained by the continuous map

$$(0, 1] \longrightarrow X \subseteq \mathbb{R}^2, \quad x \longmapsto \left( x, \sin \frac{1}{x} \right).$$

By the continuity of $f$, $X$ and hence $\overline{X}$ are connected, but $\overline{X}$ is not path-connected.

**Theorem 3.1.16.** *We have*

$$\lim_{x\to\infty} f(x) \text{ exists} \iff \begin{array}{c} \forall\, \epsilon > 0\ \exists\, M > 0 \text{ such that} \\ |f(x_1) - f(x_2)| < \epsilon \\ \text{whenever } |x_1|, |x_2| > M. \end{array} \tag{3.1.13}$$

$$\lim_{x\to a-} f(x) \text{ exists} \iff \begin{array}{c} \forall\, \epsilon > 0\ \exists\, \delta > 0 \text{ such that} \\ |f(x_1) - f(x_2)| < \epsilon \\ \text{whenever } a - \delta < x_1, x_2 < a. \end{array} \tag{3.1.14}$$

PROOF. $\Longrightarrow$: Clearly.

$\Longleftarrow$: Cauchy test for sequence $\Rightarrow \forall \{a_n\}_{n\geq 1} \to \infty$, $\{f(a_n)\}_{n\geq 1}$ converges $\Rightarrow$ $\lim_{x\to\infty} f(x)$ exists by Heine's theorem, Theorem 3.1.14. □

## 3.2. Various comparison symbols

We have proved that $\lim_{x\to 0} \sin x/x = 1$, so that in principal, we can replace $\sin x$ by $x$. A natural question is in context when/how we can do it. This section answers this question.

**3.2.1. Infinitesimals.** Actually we have six limit types:

$$x \to a, \;\; x \to a+, \;\; x \to a-, \;\; x \to -\infty, \;\; x \to +\infty, \;\; x \to \infty.$$

We will focus on the first type.

**Definition 3.2.1.** We say that $f(x)$ is an **infinitesimal** or $\boldsymbol{f(x) = o(1)}$ as $x \to a$ if $\lim_{x \to a} f(x) = 0$.

**Remark 3.2.2.** (1) $\lim_{x\to a} f(x) = A \iff f(x) - A = o(1)$ as $x \to a$.

(2) $f(x) = o(1)$ as $x \to a \iff |f(x)| = o(1)$ as $x \to a$.

(3) $f(x) = o(1), g(x) = o(1)$ as $x \to a \iff \forall \alpha, \beta \in \mathbb{R}, \alpha f(x) + \beta g(x) = o(1)$ as $x \to a$.

(4) $f(x) = o(1)$ as $x \to a$, and $g(x)$ is bounded in $\mathring{U}(a, \delta)$ (for some $\delta > 0$) $\Longrightarrow$ $f(x)g(x) = o(1)$ as $x \to a$.

**Example 3.2.3.** (1) $x \to 0$:

$$\sin x = o(1), \quad \tan x = o(1), \quad a^x - 1 = o(1) \; (a > 0).$$

$x \to 0+$:

$$x^\alpha = o(1) \; (\alpha > 0), \quad 1 - \cos x = o(1).$$

$x \to +\infty$:

$$\frac{1}{x^\alpha} = o(1) \; (\alpha > 0), \quad a^x = o(1) \; (0 < a < 1).$$

$x \to \infty$:

$$\frac{1}{x^n} = o(1) \; (n \in \mathbb{Z}_+), \quad \frac{1}{x^{1/3}} = o(1).$$

$x \to -\infty$:

$$a^x = o(1) \; (a > 1).$$

(2) $x \to 0$:

$$xe^x + 3\ln(1+x) = o(1), \quad e^{\sin x}\cos x - 1 = o(1).$$

$x \to \infty$:

$$\frac{x + \sin x}{x^2 + 5x - 2} = \frac{3x}{e^x + \ln x} = \sqrt{x+1} - \sqrt{x} = \ln\left(1 + \frac{1}{x}\right) + \frac{\sin x}{x} = o(1).$$

(3) Find $A$ and $B$ so that

$$\lim_{x\to\infty} \left(\sqrt{x^2 + 2x + 5} + Ax + B\right) = 1.$$

PROOF. Write

$$\sqrt{x^2 + 2x + 5} + Ax + B = 1 + \alpha(x), \quad \alpha(x) = o(1) \; (x \to +\infty).$$

Then

$$A = -\sqrt{1 + \frac{2}{x} + \frac{5}{x^2}} + \frac{1 - B}{x} + \frac{\alpha}{x}.$$

Letting $x \to +\infty$ yields $A = -1$ and then

$$B = 1 + \alpha + x - \sqrt{x^2 + 2x + 5}.$$

But $B$ is a constant, we must have

$$B = 1 + \lim_{x \to +\infty} \frac{-2x - 5}{\sqrt{x^2 + 2x + 5} + x} = 1 - \lim_{x \to +\infty} \frac{2 + \frac{5}{x}}{1 + \sqrt{1 + \frac{2}{x} + \frac{5}{x^2}}} = 0.$$

Thus $\sqrt{x^2 + 2x + 5} - x - 1 = o(1)$ as $x \to \infty$. □

**Definition 3.2.4.** Assume that $u(x) = o(1)$ and $v(x) = o(1)$ as $x \to a$.

(1) We say

$$u(x) = o(v(x)) \text{ as } x \to a \iff \lim_{x \to a} \frac{u(x)}{v(x)} = 0. \tag{3.2.1}$$

In particular, when $v(x) \equiv 1$, we get our old notion.

(2) We say

$$u(x) = O(v(x)) \text{ as } x \to a \iff \left|\frac{u(x)}{v(x)}\right| \le M \text{ (in some } \mathring{U}(a, \delta)). \tag{3.2.2}$$

(3) We say

$$\begin{aligned} u(x) \approx v(x) \text{ as } x \to a &\iff u(x) = O(v(x)) \text{ and } v(x) = O(u(x)) \text{ as } x \to a \\ &\iff 0 < m \le \left|\frac{u(x)}{v(x)}\right| \le M \text{ } (\exists\, 0 < m < M \text{ in } \mathring{U}(a, \delta)). \end{aligned} \tag{3.2.3}$$

(4) We say

$$u(x) \sim v(x) \text{ as } x \to a \iff \lim_{x \to a} \frac{u(x)}{v(x)} = 1. \tag{3.2.4}$$

**Proposition 3.2.5.** *(1) $u(x) = v(x) = o(1)$ as $x \to a$ and $\lim_{x \to a} u(x)/v(x)$ exists $\Longrightarrow$*

$$u(x) = O(v(x)) \quad \text{as } x \to a.$$

*(2) $u(x) = v(x) = o(1)$ as $x \to a$ and $\lim_{x \to a} u(x)/v(x) = c \neq 0 \Longrightarrow$*

$$u(x) \approx v(x) \quad \text{as } x \to a.$$

**Definition 3.2.6.** (1) We say that $u(x)$ is an **$k$-th infinitesimal** as $x \to a$ if

$$u(x) \approx (x - a)^k \quad (k > 0). \tag{3.2.5}$$

(2) We say that $c(x - a)^k$ is the **principal part of** $u(x)$ as $x \to a$ if

$$u(x) \sim c(x - a)^k \quad \text{as } x \to a. \tag{3.2.6}$$

**Example 3.2.7.** (1) $\sin x \approx (x-0)^1$, $1-\cos x \approx (x-0)^2$, $1-\cos x \approx \frac{1}{2}(x-0)^2$ as $x \to 0$.

(2) $\ln x = o(1)$ as $x \to 0+$, but

$$x^\alpha \ln x = o(1) \quad \text{as } x \to 0+ \quad (\alpha > 0,\ k \in \mathbb{Z}_+)$$

and

$$x^\alpha (\ln x)^k = o(1) \quad \text{as } x \to 0+.$$

(3) As $x \to 0$

$$\sin x \sim x \sim \tan x \sim \ln(1+x) \sim e^x - 1 \sim \frac{(1+x)^\alpha - 1}{\alpha}. \tag{3.2.7}$$

Because

$$\lim_{x\to 0}(1+x)^{1/x} = e \quad \Longrightarrow \quad \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1.$$

Letting $t = e^x - 1$ yields

$$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{t\to 0}\frac{t}{\ln(1+t)} = 1.$$

Writing

$$\frac{(1+x)^\alpha - 1}{x} = \frac{e^{\alpha\ln(1+x)}-1}{x} = \frac{e^{\alpha\ln(1+x)}-1}{\alpha\ln(1+x)}\frac{\alpha\ln(1+x)}{x}$$

we obtain $(1+x)^\alpha - 1 \sim \alpha x$.

**Proposition 3.2.8.** *We have*

*(1)* $u(x) = O(v(x))$ *and* $v(x) = O(w(x)) \Longrightarrow u(x) = O(w(x))$.
*(2)* $u(x) = O(v(x))$ *and* $v(x) = o(w(x)) \Longrightarrow u(x) = o(w(x))$.
*(3)* $O(u(x)) + O(v(x)) = O(u(x) + v(x))$.
*(4)* $O(u(x))O(v(x)) = O(u(x)v(x))$. *In particular,* $O(u(x)^k) = [O(u(x))]^k$.
*(5)* $o(1)O(u(x)) = o(u(x))$.
*(6)* $O(1)o(u(x)) = o(u(x))$.
*(7)* $O(u(x)) + o(u(x)) = O(u(x))$.
*(8)* $o(u(x)) + o(v(x)) = o(|u(x)| + |v(x)|)$.
*(9)* $o(u(x))o(v(x)) = o(u(x)v(x))$. *In particular,* $o(u(x)^k) = [o(u(x))]^k$.
*(10)* $u(x) \sim v(x)$ *and* $v(x) \sim w(x) \Longrightarrow u(x) \sim w(x)$.
*(11)* $u(x) \sim v(x)$ *and* $w(x) = o(u(x)) \Longrightarrow u(x) \sim v(x) \pm w(x)$.

### 3.2.2. Infinities.

We say $f(x)$ is an infinity if $1/f(x)$ is an infinitesimal.

**Definition 3.2.9.** Assume that $f$ is defined in $\mathring{U}(a,\rho)$. Define

(3.2.8) $$\lim_{x\to a} f(x) = +\infty \iff \begin{array}{l}\forall\, C>0\ \exists\, \delta>0,\ \forall\, x\in \mathring{U}(a,\delta)\\ \text{with } \delta<\rho,\ \text{we have } f(x)\ge C.\end{array}$$

(3.2.9) $$\lim_{x\to a} f(x) = -\infty \iff \begin{array}{l}\forall\, C>0\ \exists\, \delta>0,\ \forall\, x\in \mathring{U}(a,\delta)\\ \text{with } \delta<\rho,\ \text{we have } f(x)\le -C.\end{array}$$

(3.2.10) $$\lim_{x\to a} f(x) = \infty \iff \begin{array}{l}\forall\, C>0\ \exists\, \delta>0,\ \forall\, x\in \mathring{U}(a,\delta)\\ \text{with } \delta<\rho,\ \text{we have } |f(x)|\ge C.\end{array}$$

Similarly we can consider

$$\lim_{x\to a+},\quad \lim_{x\to a-},\quad \lim_{x\to +\infty},\quad \lim_{x\to -\infty},\quad \lim_{x\to\infty}.$$

We give several remarks:

(1) $u(x)\to\infty, v(x)\to\infty \Longrightarrow$

$$u(x)=o(v(x)) \iff \lim_{x\to a}\frac{u(x)}{v(x)}=0 \text{ or } \lim_{x\to a}\frac{v(x)}{u(x)}=\infty.$$

(2) $u(x)\to\infty, v(x)\to\infty \Longrightarrow$

$$u(x)=O(v(x)) \iff \left|\frac{u(x)}{v(x)}\right|\le M \text{ (in some } \mathring{U}(a,\delta)).$$

(3) $u(x)\to\infty, v(x)\to\infty \Longrightarrow$

$$u(x)\approx v(x) \iff u(x)=O(v(x)) \text{ and } v(x)=O(u(x)).$$

(4) $u(x)\to\infty, v(x)\to\infty \Longrightarrow$

$$u(x)\sim v(x) \iff \lim_{x\to a}\frac{u(x)}{v(x)}=1.$$

**Proposition 3.2.10.** *Proposition 3.2.8 also holds.*

**3.2.3. Equivalent substitutions.** When $x\to 0$, we proved that $\sin x\sim \tan x\sin x$, hence $\tan x-\sin x\to 0$ as $x\to 0$.

**Example 3.2.11.** Compute

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}.$$

PROOF. Observe

$$\frac{\tan x-\sin x}{x^3}=\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}=\frac{\sin x}{\cos x}\cdot\frac{1-\cos x}{x^3}=\frac{\sin x}{x}\cdot\frac{1}{\cos x}\cdot\frac{1-\cos x}{x^2}$$

which tends to $1\times 1\times\frac{1}{2}=\frac{1}{2}$.

If we use $\tan x - \sin x = o(1)$, then the limit can not be calculated. The reason is that $o(1)$ is much more coarser than $x^3$. It leads to to find more precise expression for $\tan x - \sin x$. Actually, by Taylor's expression, we have

$$\sin x - x \sim -\frac{1}{3}x^3, \quad \tan x - x \sim \frac{1}{6}x^3 \quad (\text{as } x \to 0).$$

By the remark in Example 3.2.13 (3), we have $\tan x - \sin x \sim \frac{1}{3}x^3 + \frac{1}{6}x^3 = \frac{1}{2}x^3$. □

**Theorem 3.2.12.** *$v(x) \sim w(x)$ as $x \to a$ are equivalent infinitesimal or infinity $\Longrightarrow$*

$$\lim_{x\to a} u(x)v(x) = A \iff \lim_{x\to a} u(x)w(x) = A, \tag{3.2.11}$$

$$\lim_{x\to a} \frac{u(x)}{v(x)} = A \iff \lim_{x\to a} \frac{u(x)}{w(x)} = A. \tag{3.2.12}$$

PROOF. Because $u(x)w(x) = u(x)v(x) \cdot \frac{w(x)}{v(x)}$ and $\frac{u(x)}{w(x)} = \frac{u(x)}{v(x)} \cdot \frac{v(x)}{w(x)}$. □

**Example 3.2.13.** (1) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt[3]{1+x}}{\ln(1+2x)}$.

PROOF. Because $\ln(1+2x) \sim 2x$ as $x \to 0$, we have

$$\frac{\sqrt{1+x}-\sqrt[3]{1+x}}{\ln(1+2x)} \sim \frac{\sqrt{1+x}-\sqrt[3]{1+x}}{2x} = \frac{(\sqrt{1+x}-1)-(\sqrt[3]{1+x}-1)}{2x}.$$

Using (3.2.7) yields

$$(1+x)^{1/2} - 1 \sim \frac{1}{2}x, \quad (1+x)^{1/3} - 1 \sim \frac{1}{3}x \quad (x \to 0).$$

Hence

$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt[3]{1+x}}{\ln(1+2x)} = \lim_{x\to 0} \frac{\frac{1}{2}x}{2x} - \lim_{x\to 0} \frac{\frac{1}{3}x}{2x} = \frac{1}{12}.$$

□

(2) $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt[3]{1+3x}}{x^2}$.

PROOF. From (1),

$$\sqrt{1+2x} - 1 \sim \frac{1}{2} \times 2x = x \quad \text{as } x \to 0.$$

We shall consider

$$\sqrt{1+2x} - \sqrt[3]{1+3x} = \left[\sqrt{1+2x} - (1+x)\right] - \left[\sqrt[3]{1+3x} - (1+x)\right].$$

Now

$$\lim_{x\to 0} \frac{\sqrt{1+2x}-(1+x)}{x^2} = \lim_{x\to 0} \frac{-x^2}{x^2[\sqrt{1+2x}+(1+x)]} = \lim_{x\to 0} \frac{-1}{1+x+\sqrt{1+2x}} = \frac{-1}{2}.$$

Similarly

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt[3]{1+3x}-(1+x)}{x^2} &= \lim_{x\to 0}\frac{(1+3x)-(1+x)^3}{x^2[(1+3x)^{2/3}+(1+3x)^{1/3}(1+x)+(1+x)^2]}\\ &= \frac{-3}{3} = -1.\end{aligned}$$

Finally,

$$\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt[3]{1+3x}}{x^2} = -\frac{1}{2}+1 = \frac{1}{2}.$$

□

(3) Actually, $\forall\ \alpha > 0$,

(3.2.13)

$$(1+x)^\alpha - \left[1+\sum_{1\le i\le k-1}\frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!}x^i\right] \sim \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}x^k$$

as $x \to 0$. For example,

$$\begin{aligned}(1+x)^{1/2}-\left[1+\frac{1}{2}x\right] &\sim -\frac{1}{4}x^2\\ (1+x)^{1/3}-\left[1+\frac{1}{3}x\right] &\sim -\frac{1}{9}x^2\end{aligned}$$

and

$$\begin{aligned}(1+2x)^{1/2}-[1+x] &\sim -\frac{1}{2}x^2,\\ (1+3x)^{1/3}-[1+x] &\sim -x^2\end{aligned}$$

as $x \to 0$.

In general, if

$$\begin{aligned}u(x)-\left[a_0+a_1x+\cdots+a_{k-1}x^{k-1}\right] &\sim a_kx^k,\\ v(x)-\left[a_0+a_1x+\cdots+a_{k-1}x^{k-1}\right] &\sim b_kx^k,\end{aligned}$$

with $a_k \neq b_k$, then

$$u(x)-v(x) \sim (a_k-b_k)x^k$$

as $x \to 0$.

PROOF. The proof is simple. By the assumptions,

$$\begin{aligned}\frac{u(x)-v(x)}{(a_k-b_k)x^k} &= \frac{[u(x)-P(x)]-[v(x)-P(x)]}{(a_k-b_k)x^k}\\ &\to \frac{a_k}{a_k-b_k}-\frac{b_k}{a_k-b_k} = 1\end{aligned}$$

where $P(x) := a_0+a_1x+\cdots+a_{k-1}x^{k-1}$. □

(4) $\lim_{x\to+\infty}\arccos(\sqrt{x^2+x}-x)$.

PROOF. Let $u = \sqrt{x^2+x} - x$ so that

$$\lim_{x\to+\infty} u = \lim_{x\to+\infty} \frac{x}{x+\sqrt{x^2+x}} = \frac{1}{2}.$$

Therefore

$$\lim_{x\to+\infty} \arccos(\sqrt{x^2+x} - x) = \lim_{u\to\frac{1}{2}} \arccos u = \frac{\pi}{3}.$$

□

(5) Prove $\sqrt{x+\sqrt{x+\sqrt{x}}} \sim x^{1/2}$ as $x \to +\infty$.

PROOF. We have

$$\lim_{x\to+\infty} \frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^{1/2}} = \lim_{x\to+\infty} \left[1 + \left(\frac{x+x^{1/2}}{x}\right)^{1/2}\right]^{1/2}$$

$$= \lim_{u\to 0}(1+u^{1/2})^{1/2} = 1, \quad \text{when} \quad u := \frac{x+x^{1/2}}{x^2}.$$

□

(6) Find principal parts of

$$\sin\left(x+\frac{\pi}{3}\right) - \frac{\sqrt{3}}{2}, \quad \pi - 3\arccos\left(x+\frac{1}{2}\right) \quad (x \to 0).$$

PROOF. Indeed,

$$\sin\left(x+\frac{\pi}{3}\right) - \frac{\sqrt{3}}{2} = \sin\left(x+\frac{\pi}{3}\right) - \sin\frac{\pi}{3} = 2\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)\sin\frac{x}{2} \sim \sin\frac{x}{2} \sim \frac{x}{2}$$

and

$$\pi - 3\arccos\left(x+\frac{1}{2}\right) \sim \left[\pi - 3\arccos\left(x+\frac{1}{2}\right)\right]$$

$$= \sin\left[3\arccos\left(x+\frac{1}{2}\right)\right] \qquad \left(\sin(3\theta) = 3\sin\theta - 4\sin^3\theta\right)$$

$$= 3\sqrt{1-\left(x+\frac{1}{2}\right)^2} - 4\left(\sqrt{1-\left(x+\frac{1}{2}\right)^2}\right)^3$$

$$= \sqrt{1-\left(x+\frac{1}{2}\right)^2}\left\{3 - 4\left[1-\left(x+\frac{1}{2}\right)^2\right]\right\} \sim \frac{\sqrt{3}}{2}(4x+4x^2) \sim 2\sqrt{3}x$$

as $x \to 0$. □

## 3.3. Continuities and discontinuities

# Bibliography


[1] Alarcon A., Ferrer L., Martin F., *Density theorems for complete minimal surfaces in* $\mathbb{R}^3$, Geom. Funct. Anal., **18**(2008), no. 1, 1–49.