HW2, due to 4/1/2021

Exercise 1. A map $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ between two topological spaces is **open** if $f(U) \in \mathscr{T}_Y$ for all $U \in \mathscr{T}_X$; f is **closed** if $f(C)$ is closed in Y for all closed subsets C of X.

When f is a bijective continuous map, show that the following are equivalent:

(1) f is a homeomorphism (that is, f is bijective and f, f^{-1} both are continuous).
(2) f is open.
(3) f is closed.

Exercise 2. Let $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ be a map between topological spaces. Show that

(1) f is continuous if and only if $f^{-1}(\mathbf{Int}(B)) \subseteq \mathbf{Int}(f^{-1}(B))$ for all $B \subseteq Y$.
(2) f is open if and only if $f^{-1}(\mathbf{Int}(B)) \supseteq \mathbf{Int}(f^{-1}(B))$ for all $B \subseteq Y$.

Exercise 3. On $X := \mathbf{R}$ define three subsets of 2^X:

- **(Finite complement topology)** $\mathscr{T}_{\mathbf{fc}}$ is defined by
$$\mathscr{T}_{\mathbf{fc}} := \{A \subseteq \mathbf{R} | \mathbf{R} \setminus A \text{ is finite}\} \cup \{\emptyset\}.$$
- **(Particular point topology)** $\mathscr{T}_{\mathbf{pp}}$ is defined by
$$\mathscr{T}_{\mathbf{pp}} := \{U \subseteq \mathbf{R} | U = \emptyset \text{ or } 0 \in U\}.$$
- **(Excluded point topology)** $\mathscr{T}_{\mathbf{ep}}$ is defined by
$$\mathscr{T}_{\mathbf{ep}} := \{U \subseteq \mathbf{R} | U = \mathbf{R} \text{ or } 0 \notin U\}.$$

Prove that $\mathscr{T}_{\mathbf{fc}}, \mathscr{T}_{\mathbf{pp}}, \mathscr{T}_{\mathbf{ep}}$ are all topologies on $\mathbf{R}$. Moreover, show that $(\mathbf{R}, \mathscr{T}_{\mathbf{pp}})$ and $(\mathbf{R}, \mathscr{T}_{\mathbf{ep}})$ both are first countable but not second countable, and $(\mathbf{R}, \mathscr{T}_{\mathbf{fc}})$ is neither first countable nor second countable.

Exercise 4. Let $\mathfrak{C}$ be a category. An object $P \in \mathbf{Ob}(\mathfrak{C})$ is **initial** if for any $Y \in \mathbf{Ob}(\mathfrak{C})$, $\mathbf{Hom}_{\mathfrak{C}}(P, Y)$ has exactly one element. An object $Q \in \mathbf{Ob}(\mathfrak{C})$ is **final** if for any $X \in \mathbf{Ob}(\mathfrak{C})$, $\mathbf{Hom}_{\mathfrak{C}}(X, Q)$ has exactly one element. Show that two initial (resp. final) objects are isomorphic(A morphism $f : X \to Y$ is said to be **isomorphic** if there exists $g : Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$).

Exercise 5. The **opposite category** $\mathfrak{C}^\circ$ of a category $\mathfrak{C}$ is defined by
$$\mathbf{Ob}(\mathfrak{C}^\circ) := \mathbf{Ob}(\mathfrak{C}), \quad \mathbf{Hom}_{\mathfrak{C}^\circ}(X, Y) := \mathbf{Hom}_{\mathfrak{C}}(Y, X).$$
A **contravariant functor** $G : \mathfrak{C} \to \mathfrak{C}'$ between two categories is a functor from $\mathfrak{C}^\circ$ to $\mathfrak{C}'$.

Let $\mathfrak{C}$ be a category and $X \in \mathbf{Ob}(\mathfrak{C})$ is a given object. Define
$$\mathbf{Hom}_{\mathfrak{C}}(\cdot, X) : \mathfrak{C} \longrightarrow \mathbf{Set}, \quad Z \longmapsto \mathbf{Hom}_{\mathfrak{C}}(Z, X).$$
Show that $\mathbf{Hom}_{\mathfrak{C}}(\cdot, X)$ is a contravariant functor.

Exercise 6 (Bonus). Let $(X, \mathscr{T})$ be a topological space. A **presheaf** $\mathscr{F}$ **of sets (resp. Abelian groups)** on X consists of

(a) for any nonempty open subset U of X, we have a **set (resp. Abelian group)** $\mathscr{F}(U)$,
(b) for any inclusion $V \subseteq U$ of nonemptyset open subsets of X, there exists a map $\boldsymbol{\rho}^{\mathscr{F}}_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$.

These data satisfy

(i) $\boldsymbol{\rho}^{\mathscr{F}}_{U,U} = \mathbf{1}_{\mathscr{F}(U)}$,
(ii) for any inclusions $W \subseteq V \subseteq U$ of nonempty open subsets of X, we have
$$\boldsymbol{\rho}^{\mathscr{F}}_{U,W} = \boldsymbol{\rho}^{\mathscr{F}}_{V,W} \circ \boldsymbol{\rho}^{\mathscr{F}}_{U,V}.$$

For $s \in \mathscr{F}(U)$, we usually write $s|_V := \boldsymbol{\rho}^{\mathscr{F}}_{U,V}(s)$. We define $\mathscr{F}(\emptyset) := \{0\}$.

Define $\mathfrak{C}_X$ to be the category of open subsets of X, that is,
$$\mathbf{Ob}(\mathfrak{C}_X) = \mathscr{T}, \quad \mathbf{Hom}_{\mathfrak{C}_X}(U, V) := \begin{cases} \emptyset, & \text{if } V \nsubseteq U, \\ \text{inclusion } V \hookrightarrow U, & \text{if } V \subseteq U. \end{cases}$$

Show that a presheaf $\mathscr{F}$ of sets (resp. Abelian groups) on X is a contravariant functor from $\mathfrak{C}_X$ to **Set** (reps. **AGroup**, the category of Abelian groups).

Exercise 7 (Bonus). Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces.

(1) Given an Abelian group A. For any nonempty open subset U of X, define
$$\mathscr{F}(U) := A,$$
and for any inclusion $V \subset U$ of nonempty open subsets, define
$$\boldsymbol{\rho}^{\mathscr{F}}_{U,V} := \mathbf{1}_A.$$
Show that $\mathscr{F}$ is a presheaf of Abelian groups on X, called the **constant presheaf associated to** A and denoted by $\underline{A}_X$.

(2) Given a continuous map $\boldsymbol{\pi} : (Y, \mathscr{T}_Y) \to (X, \mathscr{T}_X)$. For any nonempty open subset U of X, define
$$\mathscr{F}(U) := \left\{ \boldsymbol{s} : U \to \boldsymbol{\pi}^{-1}(U) \middle| s \text{ is continuous and } \boldsymbol{\pi} \circ \boldsymbol{s} = \mathbf{1}_U \right\},$$
and for any inclusion $V \subset U$ of nonempty open subsets, define
$$\boldsymbol{\rho}^{\mathscr{F}}_{U,V} := \text{restriction } V \hookrightarrow U.$$
Show that $\mathscr{F}$ is a presheaf of sets on X.