HW8 (Last!), due to 1/2/2020

Let M be a given smooth manifold.

1. Let $x \in M$. C^∞ Functions f and g defined on open sets containing x are said to have the same **germ** at x if they agree on some neighborhood of x. Define

$$f \sim g \quad \Leftrightarrow \quad f \text{ and } g \text{ have the same germ.}$$

Show that $\sim$ is an equivalence relation. The equivalence classes are also called **germs** and we denote the set of germs at x by $\widetilde{\mathscr{F}_x}$.

2. If f is a C^∞ function on a neighborhood of x, then $\mathbf{f}$ will denote its germ. Show that $\widetilde{\mathscr{F}_x}$ forms an algebra over $\mathbb{R}$, and $\mathbf{f}(x)$ is well-defined.

3. Let $\mathscr{F}_x \subset \widetilde{\mathscr{F}_x}$ be the set of germs which vanish at x. Show that $\mathscr{F}_x$ is an ideal in $\widetilde{\mathscr{F}_x}$. Let $\mathscr{F}_x^k$ be the ideal of $\widetilde{\mathscr{F}_x}$ consisting of all finite linear combinations of k-fold products of elements of $\mathscr{F}_x$. Show that these form a descending sequence of ideals $\widetilde{\mathscr{F}_x} \supset \mathscr{F}_x \supset \mathscr{F}_x^2 \supset \mathscr{F}_x^3 \supset \cdots$.

4. A **tangent vector** v **at the point** $x \in M$ is a linear derivation of the algebra $\widetilde{\mathscr{F}_x}$. That is, for all $\mathbf{f}, \mathbf{g} \in \widetilde{\mathscr{F}_x}$ and $\lambda \in \mathbb{R}$,

$$v(\mathbf{f} + \lambda \mathbf{g}) = v(\mathbf{f}) + \lambda v(\mathbf{g}), \quad v(\mathbf{f} \cdot \mathbf{g}) = \mathbf{f}(x) v(\mathbf{g}) + \mathbf{g}(x) v(\mathbf{f}).$$

$T_x M$ denotes the set of tangent vectors to M at x and is called the **tangent space to** M **at** x. Clearly that $T_x M$ is a real vector space.

If $\mathbf{c}$ is the germ of a function with the constant value c on a neighborhood of x, and if v is a tangent vector at x, show that $v(\mathbf{c}) = 0$.

5. Show that $T_x M$ is naturally isomorphic with $(\mathscr{F}_x / \mathscr{F}_x^2)^*$, where $*$ denotes dual vector space. (**Hint:** if $v \in T_x M$, then $v$is a linear function on $\mathscr{F}_x$ vanishing on $\mathscr{F}_x^2$. Conversely, if $\ell \in (\mathscr{F}_x / \mathscr{F}_x^2)^*$, we can define a tangent vector v_ℓ at x by setting $v_\ell(\mathbf{f}) := \ell(\{\mathbf{f} - \mathbf{f(x)}\})$ for $\mathbf{f} \in \widetilde{\mathscr{F}_x}$. Here $\mathbf{f(x)}$ denotes the germ of the function with the constant value $\mathbf{f}(x)$, and $\{\}$ is used to denote cosets in $\mathscr{F}_x / \mathscr{F}_x^2$)

***6.** Show that $\dim(\mathscr{F}_x / \mathscr{F}_x^2) = \dim M$. (**Hint:** Recall a fact from calculus: if $g \in C^k(U)$, $k \geq 2$ and U convex open about $\boldsymbol{p} \in \mathbb{U}^d$, then for each $\boldsymbol{q} \in U$,

$$\begin{aligned} g(\boldsymbol{q}) &= g(\boldsymbol{p}) + \sum_{1 \leq i \leq d} \left.\frac{\partial g}{\partial x^i}\right|_{\boldsymbol{p}} [r_i(\boldsymbol{q}) - r_i(\boldsymbol{p})] \\ &\quad + \sum_{1 \leq i,j \leq d} [r_i(\boldsymbol{q}) - r_i(\boldsymbol{p})][r_j(\boldsymbol{q}) - r_j(\boldsymbol{p})] \int_0^1 (1-t) \left.\frac{\partial^2 g}{\partial x^i \partial x^j}\right|_{(\boldsymbol{p} + t(\boldsymbol{q} - \boldsymbol{p}))} dt. \end{aligned}$$

Now, let (U, φ) be a coordinate chart of x with coordinate functions $x^1, \cdots, x^d$, where $d = \dim M$. Let $\mathbf{f} \in \mathscr{F}_x$. Apply the above formula to $f \circ \varphi^{-1}$ and compose with φ to obtain

$$f = \sum_{1 \leq i \leq d} \left.\frac{(\partial (f \circ \varphi^{-1})}{\partial x^i}\right|_{\varphi(x)} [x^i - x^i(x)] + \sum_{1 \leq i,j \leq d} [x^i - x^i(x)][x^j - x^j(x)] h$$

on a neighborhood of x, where $h \in C^\infty$. Hence $\{\{\mathbf{x}^i - \mathbf{x}^i(\mathbf{x})\}\}_{1 \leq i \leq d}$ spans $\mathscr{F}_x / \mathscr{F}_x^2$. On the other hand, suppose

$$\sum_{1 \leq i \leq d} a_i \left(\mathbf{x}^i - \mathbf{x}^i(\mathbf{x})\right) \in \mathscr{F}_x^2.$$

Then $\frac{\partial}{\partial x^j}|_{\varphi(x)} [\sum_{1 \leq i \leq d} a^i (x^i - x^i(\varphi(x)))] = 0$.)