HW1, due to 10/12/2020

Exercise 1. Given a metric space (X, d) we define a relation $\sim$ on X by

$$x \sim y \Longleftrightarrow d(x, y) < \infty.$$

Then $\sim$ is an equivalence relation on X, and the equivalence class $[x]$ of x can be endowed with a natural metric (still denoted by d). For any $x \in X$, show that $([x], d)$ is a finite metric space.

PROOF. Clearly $x \in [x]$ and $\sim$ is an equivalence relation on X. For any $y_1, y_2 \in [x]$ we have $y_1 \sim x$ and $y_2 \sim x$, so that

$$d(y_1, y_2) \leq d(y_1, x) + d(x, y_2) < +\infty.$$

Thus d is finite. □

Exercise 2. Given a semi-metric space (X, d) we define a relation $\sim$ on X by

$$x \sim y \Longleftrightarrow d(x, y) = 0.$$

Show that $\sim$ is an equivalence relation on X. Define

$$\hat{X} := X/\sim = \{[x] : x \in X\}, \quad \hat{d}([x], [y]) := d(x, y).$$

Show that $(X/d, d) := (\hat{X}, \hat{d})$ is a metric space.

PROOF. If $x' \sim x$ and $y' \sim y$, then

$$d(x', y') \leq d(x', x) + d(x, y) + d(y, y') = d(x, y) \leq d(x, x') + d(x', y') + d(y', y) = d(x', y').$$

Thus the map $\hat{d}$ is well-defined.

If $\hat{d}([x], [y]) = 0$, then $d(x, y) = 0$ and $x \sim y$. Hence $[x] = [y]$. The triangle inequality for $\hat{d}$ follows immediately from d. □

Exercise 3. Let $X = \mathbf{R}^2$ and define

$$d((x, y), (x', y')) := |(x - x') + (y - y')|.$$

Show that d is a semi-metric on X. Define $\boldsymbol{f} : \mathbf{R}^2/d \to \mathbf{R}$ by $\boldsymbol{f}([(x, y)]) := x + y$. Show that $\boldsymbol{f}$ is an isometry.

PROOF. Clearly d is a semi-metric on X. Since $d((1, 2), (2, 1)) = 0$, it follows that d is not a metric. For any $(x_1, y_1), (x_2, y_2) \in \mathbf{R}^2$ we have

$$\begin{aligned} d([(x_1, y_1)], [(x_2, y_2)]) &= d((x_1, y_1), (x_2, y_2)) = |(x_1 - x_2) + (y_1 - y_2)| \\ &= |x_1 + y_1 - (x_2 + y_2)| = d_{\mathbf{R}}(x_1 + y_1, x_2 + y_2) = d_{\mathbf{R}}(\boldsymbol{f}([(x_1, y_1)]), \boldsymbol{f}([(x_2, y_2)])). \end{aligned}$$

Thus $\boldsymbol{f}$ is isometric. □

Exercise 4. Consider the set X of all continuous real-valued functions on $[0, 1]$. Show that

$$d(f, g) := \int_0^1 |f(x) - g(x)| dx$$

defines a metric on X. Is this still the case if continuity is weakened to integrability?

PROOF. If $d(f, g) = 0$, then $f = g$ by continuity. The triangle inequality is clear. Hence d is a metric. But for $f, g \in L^1([0, 1])$, $d(f, g) = 0$ may not imply $f = g$, because f and g can differ from a set of measure zero. □

Exercise 5. Let (X, d_X) and (Y, d_Y) be two metric spaces. We define

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) := \sqrt{d_X(x_1, x_2) + d_Y(y_1, y_2)}.$$

Show that $(X \times Y, d_{X\times Y})$ is a metric space.

PROOF. Obviously. □

Exercise 6. As a subspace of $\mathbf{R}^2$, the unit circle $\mathbf{S}^1$ carries the restricted Euclidean metric from $\mathbf{R}^2$. We can define another (intrinsic) metric d_{int} by

$$d_{\text{int}}(\boldsymbol{x}, \boldsymbol{y}) := \text{the length of the shorter arc between them.}$$

Note that $d_{\text{int}}(\boldsymbol{x}, \boldsymbol{y}) \in [0, \pi]$ for any points $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{S}^1$.

(a) Show that any circle arc of length less than or equal to π, equipped with the intrinsic metric, is isometric to a straight line segment.
(b) The whole circle with the intrinsic metric is not isometric to any subset of $\mathbf{R}^2$.

PROOF. □

PROOF. (a) Denote by $\sphericalangle(\boldsymbol{x}, \boldsymbol{y}) \in [0, \pi]$ the angle formed by segments $\overline{\mathbf{0}\boldsymbol{x}}$ and $\overline{\mathbf{0}\boldsymbol{y}}$ at $\mathbf{0}$. Then

$$d_{\text{int}}(\boldsymbol{x}, \boldsymbol{y}) = \sphericalangle(\boldsymbol{x}, \boldsymbol{y}).$$

For any $\boldsymbol{z}$ lying on the circle arc $\boldsymbol{x}\mathbf{0}\boldsymbol{y}$, define $f(\boldsymbol{z}) := \sphericalangle(\boldsymbol{x}, \boldsymbol{z}) =: \theta_{\boldsymbol{z}} \in [0, \theta] \subset \mathbf{R}$. Then for such two points $\boldsymbol{z}, \boldsymbol{w}$ we have

$$d_{\mathbf{R}}(f(z), f(w)) = |\theta_{\boldsymbol{z}} - \theta_{\boldsymbol{w}}| = |\sphericalangle(\boldsymbol{z}, \boldsymbol{w})| = d_{\text{int}}(\boldsymbol{z}, \boldsymbol{w}).$$

Hence f is isometric.

(b) Suppose there is an isometry $\boldsymbol{f}$ from $(\mathbf{S}^1, d_{\text{int}})$ into $(U, d_{\mathbf{R}^2})$, where$U$ is some subset of $\mathbf{R}^2$. For any $\boldsymbol{x} \in \mathbf{S}^1$, we have

$$|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(-\boldsymbol{x})| = d_{\mathbf{int}}(\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{f}(-\boldsymbol{x})) = d_{\text{int}}(\boldsymbol{x}, -\boldsymbol{x}) = \pi.$$

On the other hand, define

$$\boldsymbol{g}(\boldsymbol{x}) := \boldsymbol{f}(x) - \boldsymbol{f}(-x), \quad \boldsymbol{x} \in \mathbf{S}^1.$$

Then $\boldsymbol{g}(-\boldsymbol{x}) = -\boldsymbol{g}(\boldsymbol{x})$. By intermediate value theorem, there exists $\boldsymbol{x}_0 \in \mathbf{S}^1$ such that $\boldsymbol{g}(\boldsymbol{x}_0) = 0$; thus $\boldsymbol{f}(\boldsymbol{x}_0) = \boldsymbol{f}(-\boldsymbol{x}_0)$. Hence

$$\pi = |\boldsymbol{f}(\boldsymbol{x}_0) - \boldsymbol{f}(-\boldsymbol{x}_0)| = 0.$$

This contradiction shows that such an $\boldsymbol{f}$ does not exist. □

Exercise 7. Let $(X, \mathscr{T})$ be a topological space. A **neighborhood** of $x \in X$ is a set $N \subseteq X$ containing an open subset $U \in \mathscr{T}$ with the property $x \in U \subset N$.

Let $(X, \mathscr{T})$ be a topological space and $A \subseteq X$ be any subset. It is easy to prove the following statements:

(1) A point is in $\mathbf{Int}(A)$ if and only if it has a neighborhood contained in A.
(2) A point is in $\mathbf{Ext}(A)$ if and only if it has a neighborhood contained in $X \setminus A$.
(3) A point is in $\mathbf{Bdy}(A)$ if and only if every neighborhood of it contains both a point of A and a point if $X \setminus A$.

(4) A point is in $\overline{A}$ if and only if every neighborhood of it contain a point of A.
(5) $\overline{A} = A \cup \partial A = \mathring{A} \cup \partial A$.
(6) $\mathbf{Int}(A)$ and $\mathbf{Ext}(A)$ are open in X, while $\overline{A}$ and ∂A are closed in X.
(7) The following are equivalent:
  - A is open in X.
  - $A = \mathbf{Int}(A)$.
  - A contains none of its boundary points.
  - Every point of A has a neighborhood contained in A.

(8) The following are equivalent:
  - A is closed in X.
  - $A = \overline{A}$.
  - A contains all of its boundary points.
  - Every point of $X \setminus A$ has a neighborhood contained in $X \setminus A$.

(9) $\overline{X \setminus A} = X \setminus \mathring{A}$ and $(X \setminus A)^\circ = X \setminus \overline{A}$.
(10) Let $\mathscr{A}$ be a collection of subsets of X. Show that

$$\overline{\bigcap_{A\in\mathscr{A}}} \subseteq \bigcap_{A\in\mathscr{A}} \overline{A}, \quad \overline{\bigcup_{A\in\mathscr{A}} A} \supseteq \bigcup_{A\in\mathscr{A}} \overline{A}$$

and

$$\left(\bigcap_{A\in\mathscr{A}} A\right)^\circ \subseteq \bigcap_{A\in\mathscr{A}} \mathring{A}, \quad \left(\bigcup_{A\in\mathscr{A}} A\right)^\circ \supseteq \bigcup_{A\in\mathscr{A}} \mathring{A}.$$

PROOF. Omit. □

Exercise 8. Suppose that $(X, \mathscr{T})$ is a topological space and $A \subseteq X$ is a subset.

- We say that $x \in X$ is a **limit point (or accumulation point, cluster point) of A** if every neighborhood of x contains a point of A other than x (note that x itself may not be in A).
- We say that $x \in X$ is an **isolated point of A** if x has a neighborhood U in X such that $U \cap X = \{x\}$.
- A is said to be **dense in X** if $\overline{A} = X$.

Clearly ever point of A is either a limit point or an isolated point, but not both.

Let $(X, \mathscr{T})$ be a topological space and $A \subseteq X$. Show that

- A is closed if and only if it contains all of its limit points.
- A is dense if and only if every nonempty open subset of X contains a point of A.

PROOF. Omit. □

Exercise 9 (Bonus). A **(commutative) ring** A is a set with two binary operations $(+, \cdot)$ such that

1) $(A, +)$ is an Abelian group (so that A has an 0 element, and for any $x \in A$ there is a unique (additive) inverse $-x \in A$).
2) $x \cdot (y+z) = x \cdot y + x \cdot z$, $(y+z) \cdot x = y \cdot x + z \cdot x$, and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for any $x, y, z \in A$.
3) $x \cdot y = y \cdot x$ for any $x, y \in A$.

4) There exists a unique element $1 \in A$ such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in A$.

A map $f : A \to B$ between two rings is said to be a **ring homomorphism** if it satisfies

1) $f(x+y) = f(x) + f(y)$, for all $x, y \in A$.
2) $f(x \cdot y) = f(x) \cdot f(y)$, for all $x, y \in A$.
3) $f(1) = 1$.

Assume that A is a ring.

(1) An **ideal** $\mathfrak{a}$ of A is a subset of A which is an additive subgroup and is such that $A\mathfrak{a} \subseteq \mathfrak{a}$ (i.e., $xy \in \mathfrak{a}$ for all $x \in A$ and $y \in \mathfrak{a}$). Then $A/\mathfrak{a}$ is a ring and $\phi : A \to A/\mathfrak{a}$, $x \mapsto x + \mathfrak{a}$, is a ring homomorphism.

(2) An ideal $\mathfrak{p}$ in A is **prime** if $\mathfrak{p} \neq A$ and if $(xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ or $y \in \mathfrak{p})$.

If $f : A \to B$ is a ring homomorphism and $\mathfrak{q}$ is a prime ideal in B, then $f^{-1}(\mathfrak{q})$ is a prime ideal in A.

Let

$$X := \{\text{primes ideals of } A\}, \quad V(\mathfrak{a}) := \{\mathfrak{p} \in X | \mathfrak{a} \subseteq \mathfrak{p}\}$$

where $\mathfrak{a}$ is an ideal.

(a) Prove that

- $V(0) = X$, $V(A) = \emptyset$.
- $\mathfrak{a} \subseteq \mathfrak{b} \Rightarrow V(\mathfrak{a}) \supseteq V(\mathfrak{b})$.
- $V(\sum_{i \in I} \mathfrak{a}_i) = \cap_{i \in I} V(\mathfrak{a}_i)$ for any family of ideals $\mathfrak{a}_i$, $i \in I$, of A.
- $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$.

Here $\mathfrak{a}\mathfrak{b}$ is the ideal generated by all products $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$, that is, $\mathfrak{a}\mathfrak{b} = \{\sum_{1 \le i \le n} x_i y_i : x_i \in \mathfrak{a},\ y_i \in \mathfrak{b}\}$.

We then obtain the **Zariski topology** $(X, \mathscr{T}) =:$ Spec(A) on X, where

$$\mathscr{T} := \{U \in X : X \setminus U = V(\mathfrak{a}) \text{ for some ideal } \mathfrak{a} \text{ of } A\}.$$

(b) We have $X = \emptyset$ if and only if $0 = 1$ in A.

(c) For any ideal $\mathfrak{a}$ of A, define the **nilpotent radical** of $\mathfrak{a}$ to be the ideal

$$\sqrt{\mathfrak{a}} := \{a \in A : a^n \in \mathfrak{a} \text{ for some positive integer } n\}.$$

Then we have $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

(d) $\sqrt{\mathfrak{a}} = \cap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$.

(e) For any ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A, we have $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$.

PROOF. (a) $V(0) = X$ and $V(A) = \emptyset$ is clear. Suppose $\mathfrak{a} \subseteq \mathfrak{b}$. For any $\mathfrak{p} \in V(\mathfrak{b})$ we have $\mathfrak{b} \subseteq \mathfrak{p}$ so that $\mathfrak{a} \subseteq \mathfrak{p}$ and hence $\mathfrak{p} \in V(\mathfrak{a})$.

For any family of ideals $\mathfrak{a}_i$, we have $\mathfrak{a}_i \subseteq \sum_{i \in I} \mathfrak{a}_i$ so that $V(\sum_{i \in I} \mathfrak{a}_i) \subseteq \cap_{i \in I} V(\mathfrak{a}_i)$. On the other hand, for any $\mathfrak{p} \in \cap_{i \in I} V(\mathfrak{a}_i)$, we have $\mathfrak{a}_i \subseteq \mathfrak{p}$ for any $i \in I$. Therefore $\sum_{i \in I} \mathfrak{a}_i \subseteq \mathfrak{p}$ so $p \in V(\sum_{i \in I} \mathfrak{a}_i)$.

Since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}, \mathfrak{b}$ we have

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b}).$$

On the other hand, for any $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$ we have either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$ since $\mathfrak{p}$ is prime. Hence $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.

(b) If $0 = 1$ in A, then any idea $\mathfrak{a}$ must contain 1 and then $A = (1)$. Hence $X = \emptyset$. Suppose now that $0 \neq 1$. Let

$$\mathscr{S} := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of } A \text{ and } 1 \notin \mathfrak{a}\}.$$

Then $(0) \in \mathfrak{S}$. Since any totally ordered subset of $\mathscr{S}$ with respect to the order defined by inclusion has an upper bound in $\mathscr{S}$, by Zorn's lemma, $\mathscr{S}$ has a maximal element, that is, $\mathscr{S}$ has a maximal ideal. Any maximal ideal is prime. So X is nonempty.

(c) Because $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$, we have $V(\sqrt{\mathfrak{a}}) \subseteq V(\mathfrak{a})$. Conversely, for any $\mathfrak{p} \in V(\mathfrak{a})$ we have $\mathfrak{a} \subset \mathfrak{p}$. For any $a \in \sqrt{\mathfrak{a}}$ we have $a^n \in \mathfrak{a}$ for some positive integer n. Hence $a^n \in \mathfrak{p}$ and then $a \in \mathfrak{p}$. Thus $\sqrt{\mathfrak{a}} \subseteq \mathfrak{p}$ and then $p \in V(\sqrt{\mathfrak{a}})$.

(d) For any $\mathfrak{p} \subset V(\mathfrak{a})$, by (c) we have $\mathfrak{p} \in V(\sqrt{\mathfrak{a}})$ and $\sqrt{\mathfrak{a}} \subseteq \mathfrak{p}$. Hence $\sqrt{\mathfrak{a}} \subseteq \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$.

Suppose $a \notin \sqrt{\mathfrak{a}}$. Then in $(A/\mathfrak{a})_a$ we have $0 \neq 1$. By (b), we can find a prime ideal $\mathfrak{q}$ of $(A/\mathfrak{a})_a$. Let $\mathfrak{p}$ be the inverse image of $\mathfrak{q}$ under the canonical homomorphism $A \to (A/\mathfrak{a})_a$. Then $\mathfrak{p} \in V(\mathfrak{a})$ but $a \notin \mathfrak{p}$. Hence $a \notin \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$.

(e) By (c) and (a). $\square$