DIFFERENTIAL MANIFOLDS: HW1 (DUE TO 2021/3/24)

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Remark: Please solve problems as many as you can!

1. Let $T(m,n)$ be the space of all $m \times n$ real matrices. Then $T(m,n)$ can be regarded as $\mathbf{R}^{mn}$ and therefore is a real analytic (C^ω) manifold. Let $T(m,n;k)$ denote the space of all $m \times n$ real matrices of rank k (where $0 < k \le \min(m,n)$) with the induced topology of $T(m,n)$. Then $T(m,n;k)$ is a real analytic manifold of dimension $k(m+n-k)$. In fact, let $X_0 \in T(m,n)$. If $\operatorname{rank} X_0 \ge k$, there are permutation matrices P and Q such that

$$PX_0Q = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$$

where A_0 is a nonsingular $k \times k$ matrix. There is an $\epsilon > 0$ (depending on A_0) such that if $||A - A_0||_{\text{matrix}} < \epsilon$, then A is nonsingular.

Please complete the above proof that $T(m,n;k)$ is a real analytic manifold of dimension $k(m+n-k)$.

2. (Loomis–Whitney, 1949) Follow the steps below to prove the **isoperimetric inequality**. For $m \ge 2$ and $1 \le j \le m$ define the projection maps $\pi_j : \mathbf{R}^m \to \mathbf{R}^{m-1}$ by setting for $\boldsymbol{x} = (x^1, \cdots, x^m)$,

$$\pi_j(\boldsymbol{x}) = (x^1, \cdots, x^{j-1}, x^{j+1}, \cdots, x^m),$$

with the obvious interpretations when $j = 1$ or $j = m$.

(a) For maps $f_j : \mathbf{R}^{m-1} \to \mathbf{C}$, $j = 1, \cdots, m$, prove that

$$\Lambda(f_1, \cdots, f_n) := \int_{\mathbf{R}^n} \prod_{1 \le j \le m} |f_j \circ \pi_j| d\boldsymbol{x} \le \prod_{1 \le j \le m} ||f_j||_{L^{m-1}(\mathbf{R}^{m-1})}.$$

(b) Let Ω be a compact set with a rectifiable boundary in $\mathbf{R}^m$ where $m \ge 2$. Show that there is a constant c_m independent of Ω such that

$$|\Omega| \le c_m |\partial\Omega|^{\frac{m}{m-1}},$$

where the expression $|\partial\Omega|$ denotes the $(m-1)$-dimensional surface measure of the boundary of Ω.

3. (Poincaré upper half plane) The Poincaré upper half plane is the set

$$\mathfrak{H} := \{z \in \mathbf{C} : \operatorname{Im} z > 0\}.$$

For any commutative ring R (e.g., $\mathbf{R}, \mathbf{Z}, \mathbf{Z}/N\mathbf{Z}$), the general linear group $\mathbf{GL}_2(R)$ is defined to be the set

$$\mathbf{GL}_2(R) := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det M = ad - bc \in R^* \right\}$$

where R^* is the multiplicative group of invertible elements of R. The special linear group $\mathbf{SL}_2(R)$ is defined to be the subgroup of $\mathbf{GL}_2(R)$ consisting of matrices of determinant 1.

(a) Show that $\mathbf{GL}_2(R)$ is a group and $\mathbf{SL}_2(R)$ is a subgroup of $\mathbf{GL}_2(R)$.

Let $\overline{\mathbf{C}}$ denote the extended complex plane $\mathbf{C} \cup \{\infty\}$ (topologically isomorphic to $\mathbf{S}^2$), or equivalently the complex projective plane $\mathbf{CP}^1$. Given an element $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbf{R})$ and a point $z \in \mathbf{C}$, we define

$$Mz := \frac{az+b}{cz+d}, \quad M\infty := \frac{a}{c} = \lim_{z \to \infty} Mz.$$

Thus $M\frac{-d}{c} = \infty$, and if $c = 0$ then $M\infty = \infty$. These maps $z \mapsto Mz$ are called fractional linear transformations of the Riemann sphere $\overline{\mathbf{C}}$.

(b) Prove that fractional linear transformations define a group action on the set $\overline{\mathbf{C}}$; that is,

$$M_1(M_2 z) = (M_1 M_2) z$$

for all $M_1, M_2 \in \mathbf{SL}_2(\mathbf{R})$ and $z \in \overline{\mathbf{C}}$.

(c) By part (b), we have a group action

$$\mathbf{SL}_2(\mathbf{R}) \times \overline{\mathbf{C}} \longrightarrow \overline{\mathbf{C}}, \quad (M, z) \mapsto Mz.$$

For any $M \in \mathbf{SL}_2(\mathbf{R})$, set

$$\mathbf{SL}_2(\mathbf{R})_{\text{fix}} := \left\{ M \in \mathbf{SL}_2(\mathbf{R}) : Mz = z \text{ for all } z \in \overline{\mathbf{C}} \right\}.$$

Show that $\mathbf{SL}_2(\mathbf{R})_{\text{fix}}$ is a subgroup of $\mathbf{SL}_2(\mathbf{R})$ and is equal to $\{\pm I_2\}$, where I_2 denotes the 2×2 identity matrix. The quotient group $\mathbf{PSL}_2(\mathbf{R}) := \mathbf{SL}_2(\mathbf{R})/\mathrm{SL}_2(\mathbf{R})_{\text{fix}}$ is called the projective special linear group.

(d) Prove that $\mathbf{SL}_2(\mathbf{R})$ is a group action on $\mathfrak{H}$. In particular,

$$\mathrm{Im}(Mz) = \frac{1}{|cz+d|^2} \mathrm{Im}(z), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We temporarily let

$$z' := Mz = \frac{az+b}{cz+d}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $z' = x' + \sqrt{-1} y'$ and $z = x + \sqrt{-1} y$, then

$$dz' = \frac{dz}{(cz+d)^2}.$$

(e) Use part (d) to show

$$\frac{dx'^2 + dy'^2}{y'^2} = \frac{dx^2 + dy^2}{y^2}.$$

We call

$$g := \frac{dx^2 + dy^2}{y^2}$$

the **symplectic metric**. From the part (e), we know that the symplectic metric is invariant under $\mathbf{SL}_2(\mathbf{R})$. The metric g induces the symplectic length element

$$dL_g := \frac{\sqrt{dx^2 + dy^2}}{y}$$

and the symplectic area element

$$dA_g := \frac{dxdy}{y^2}.$$

Given two points $z_1, z_2 \in \mathfrak{H}$, and a smooth curve C connecting those two points, we consider the length of C

$$L_g(C) := \int_C dL_g = \int_C \frac{\sqrt{dx^2 + dy^2}}{y}.$$

Without loss of generality, we may assume that z_1, z_2 do not both lie on a line perpendicular to x axis. Hence, we can always find a circle $\bigcirc$, centered at $(t, 0)$, passing through z_1, z_2. Write $\bigcirc$ as

$$z = x + \sqrt{-1}y, \quad x = t + \rho\cos\theta, \quad y = \rho\sin\theta,\ \theta \in (0, \pi),$$

where ρ stands for the radius of $\bigcirc$. Suppose that $\theta = \theta_i$ when $z = z_i$ for $i = 1, 2$, and $0 < \theta_1 < \theta_2 < \pi$. Choose a curve C with the equation

$$x = t + \rho(\theta)\cos\theta, \quad y = \rho(\theta)\sin\theta.$$

Note that $\rho(\theta_i) = \rho$ for $i = 1, 2$.

(f) Show that

$$L_g(C) = \int_{\theta_1}^{\theta_2} \sqrt{1 + \left(\frac{\rho'(\theta)}{\rho(\theta)}\right)^2} \frac{d\theta}{\sin\theta} \geq \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sin\theta} = \ln\frac{\tan\frac{\theta_2}{2}}{\tan\frac{\theta_1}{2}}$$

and the equality holds if and only if $\rho(\theta) \equiv \rho$.

The above exercise shows that minimizing curves (or geodesics) of $L_g(\cdot)$ are half circles contained in $\mathfrak{H}$ and centered on x-axis. For any two points $z_1, z_2 \in \mathfrak{H}$, we define the symplectic distance

$$d_g(z_1, z_2) := \min_C L_g(C)$$

where the minimum is taken over all smooth curve connecting those two points.

Given three points $A, B, C \in \mathfrak{H}$, we then obtain three geodesics $\overline{AB}, \overline{BC}$, and $\overline{AC}$. The triangle enclosed by those three geodesics is called the symplectic triangle $\triangle ABC$ of A, B, C. The area of the symplectic triangle $\triangle ABC$ is given by

$$A_g(\triangle ABC) = \iint_{\triangle ABC} \frac{dxdy}{y^2}.$$

Let

$$A=(x_1,y_1),\quad B=(x_2,y_2),\quad C=(x_3,y_3),$$

and the geodesic of AC (resp., AB, BC) be centered at $(\lambda_1,0)$ (resp., $(\lambda_2,0)$, $(\lambda_3,0)$) with radius r_1 (resp., r_2,r_3). Then

$$\iint_{\triangle ABC}\frac{dxdy}{y^2}=\int_{x_1}^{x_3}dx\int_{\sqrt{r_1^2-(x-\lambda_1)^2}}^{\sqrt{r_2^2-(x-\lambda_2)^2}}\frac{dy}{y^2}+\int_{x_1}^{x_2}\int_{\sqrt{r_3^2-(x-\lambda_3)^2}}^{\sqrt{r_2^2-(x-\lambda_2)^2}}\frac{dy}{y^2}.$$

(g) Show that

$$A_g(\triangle ABC)=\pi-\angle A-\angle B-\angle C.$$

In particular, $0\le\angle A+\angle B+\angle C<\pi$.

(h) Compute the symplectic area of the symplectic triangle enclosed by

$$x^2+y^2=1,\quad (x-1)^2+y^2=1,\quad (x+1)^2+y^2=1$$

in $\mathfrak{H}$.

(i) Compute the symplectic area of the domain

$$\mathscr{D}:=\left\{(x,y)\in\mathbf{R}^2: x^2+y^2\ge 1,\ -\frac{1}{2}\le x\le\frac{1}{2},\ y>0\right\}.$$

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