HW3, due to 4/14/2021

Exercise 1. A T_2-space is T_4 if and only if for any open set U and closed set A with $A \subseteq U$, there exists an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

Exercise 2. Show that $(\mathbf{R}, \mathscr{T}_{\mathbf{pp}})$ is not Lindelöf but $(\mathbf{R}, \mathscr{T}_{\mathbf{ep}})$ is Lindelöf. Here for the definitions of $\mathscr{T}_{\mathbf{pp}}$ and $\mathscr{T}_{\mathbf{ep}}$, see HW2.

Exercise 3 (Bonus). Recall the definition of presheaves in HW2. A presheaf $\mathscr{F}$ of sets (resp. Abelian groups) is a **sheaf** if it satisfies the following conditions:

(iii) Let $s, t \in \mathscr{F}(U)$. If there is an open cover $\{U_i\}_{i\in I}$ of U such that $s|_{U_i} = t|_{U_i}$, then $s = t$;

(iv) If $\{U_i\}_{i\in I}$ is an open cover of U and $s_i \in \mathscr{F}(U_i)$ are some sections satisfying $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then there exists a section $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$.

Show that a presheaf $\mathscr{F}$ of Abelian groups is a sheaf if and only if for any open cover $\{U_i\}_{i\in I}$ of any open subset $U \subseteq X$, the sequence

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\ \alpha\ } \prod_{i\in I} \mathscr{F}(U_i) \xrightarrow{\ \beta\ } \prod_{(i,j)\in I\times I} \mathscr{F}(U_i \cap U_j)$$

is exact, that is, α is an injective homomorphism and $\mathbf{Ker}(\beta) = \mathbf{Im}(\alpha)$, where

$$\alpha(s) := \prod_{i\in I} s|_{U_i}, \quad \beta\left(\prod_{i\in I} s_i\right) = \prod_{(i,j)\in I\times I} \left(s_j|_{U_i\cap U_j} - s_i|_{U_i\cap U_j}\right).$$