DIFFERENTIAL MANIFOLDS: HW1 (DUE TO 2021/4/9)

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Remark: Please solve problems as many as you can!

1. Let $(\mathcal{M}, g)$ be a Riemannian manifold and define the **Weyl-Schouten tensor field** by

$$S_g := \mathrm{Rc}_g - \frac{R_g}{2(m-1)} g, \quad m := \dim \mathcal{M}.$$

The **Cotton tensor field** C_g is given by

$$C_{ijk} := \nabla_i S_{jk} - \nabla_j S_{ik}.$$

Show that if $m = 3$, then $C_{e^u g} = e^u C_g$ for any $u \in C^\infty(\mathcal{M})$.

2. Hamilton's cigar soliton or **Witten's black hole** is the complete Riemannian surface $(\mathbf{R}^2, g)$, where

$$g = \frac{dx \otimes dx + dy \otimes dy}{1 + x^2 + y^2}.$$

Show that the scalar curvature R_g of g is

$$R_g = \frac{dr \otimes dr + r^2 d\theta \otimes d\theta}{1 + r^2}, \quad x = r\cos\theta, \quad y = r\sin\theta.$$

3. Suppose that G is a 3-dimensional unimodular (i.e., its volume form is bi-invariant) Lie group with a left-invariant metric g. Then there exists a left-invariant frame field $\{f_i\}_{1 \le i \le 3}$, called the **Milnor frame**, with dual coframe field $\{\eta_i\}_{1 \le i \le 3}$ such that there are positive constants A, B, C such that the metric g is diagonal

$$g = A\eta^1 \otimes \eta^1 + B\eta^2 \otimes \eta^2 + C\eta^3 \otimes \eta^3$$

and the Lie brackets are of the form

$$[f_i, f_j] = c_{ij}^k f_k,$$

where $c_{ij}^k \in \{-2, 0, 2\}$ and $c_{ij}^k = 0$ unless i, j, k are distinct. Let

$$\lambda := c_{23}^1, \quad \mu := c_{31}^2, \quad \nu := c_{21}^3.$$

Show that

(a) The frame field $\{e_i\}_{1 \le i \le 3}$ defined by

$$e_1 := A^{-1/2} f_1, \quad e_2 := B^{-1/2} f_2, \quad e_3 := C^{-1/2} f_3$$

is orthonormal.

(b) Letting $\lambda_1 := A$, $\lambda_2 := B$, and $\lambda_3 := C$, then

$$\begin{aligned}\mathrm{Ric}_g(e_1, e_1) &= \frac{(\lambda A)^2 - (\mu B - \nu C)^2}{2ABC},\\ \mathrm{Ric}_g(e_2, e_2) &= \frac{(\mu B)^2 - (\nu C - \lambda A)^2}{2ABC},\\ \mathrm{Ric}_g(e_3, e_3) &= \frac{(\nu C)^2 - (\lambda A - \mu B)^2}{2ABC}.\end{aligned}$$

(c) The scalar curvature R_g is

$$R_g = -\frac{(\lambda A)^2 + (\mu B)^2 + (\nu C)^2}{2ABC} + \frac{\mu\nu BC + \nu\lambda CA + \lambda\mu AB}{ABC}.$$

(d) Show that the **normalized Ricci flow equation** $\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + \frac{2}{3}R_{g(t)}g(t)$, where

$$g(t) = A(t)\eta^1 \otimes \eta^1 + B(t)\eta^2 \otimes \eta^2 + C(t)\eta^3 \otimes \eta^3,$$

on G is equivalent to the following system

$$\begin{aligned}\frac{dA(t)}{dt} &= \frac{-4(\lambda A(t))^2 + 2(\mu B(t))^2 + 2(\nu C(t))^2}{3B(t)C(t)}\\ &+ \frac{-4\mu B(t)\nu C(t) + 2\nu C(t)\lambda A(t) + 2\lambda A(t)\mu B(t)}{3B(t)C(t)},\\ \frac{dB(t)}{dt} &= \frac{2(\lambda A(t))^2 - 4(\mu B(t))^2 + 2(\nu C(t))^2}{3A(t)C(t)}\\ &+ \frac{2\mu B(t)\nu C(t) - 4\nu C(t)\lambda A(t) + 2\lambda A(t)\mu B(t)}{3A(t)C(t)},\\ \frac{dC(t)}{dt} &= \frac{2(\lambda A(t))^2 + 2(\mu B(t))^2 - 4(\nu C(t))^2}{3A(t)B(t)}\\ &+ \frac{2\mu B(t)\nu C(t) + 2\nu C(t)\lambda A(t) - 4\lambda A(t)\mu B(t)}{3A(t)B(t)}..\end{aligned}$$

(e) **(Ricci flow on SU(2))** In this case $\lambda = \mu = \nu = -2$, and, since the normalized Ricci flow preserves volumes, we may assume

$$A(t)B(t)C(t) = \frac{8}{3}.$$

On $\mathbf{SU}(2)$ the normalized Ricci flow becomes

$$\begin{aligned}\frac{dA(t)}{dt} &= A(t)\left\{A(t)[B(t) + C(t) - 2A(t)] + [B(t) - C(t)]^2\right\},\\ \frac{dB(t)}{dt} &= B(t)\left\{B(t)[A(t) + C(t) - 2B(t)] + [A(t) - C(t)]^2\right\},\\ \frac{dC(t)}{dt} &= C(t)\left\{C(t)[A(t) + B(t) - 2C(t)] + [A(t) - B(t)]^2\right\}.\end{aligned}$$

We may without loss of generality that

$$A(0) \geq B(0) \geq C(0).$$

Show that $A(t) \geq B(t) \geq C(t) \geq C(0)$ for all $t \geq 0$.

(f) Under assumptions in (e), show that

$$A(t) - C(t) \leq [A(0) - C(0)]e^{-2C(0)^2 t}, \quad t > 0.$$

Show that $g(t)$ exponentially converges, as $t \to +\infty$, to the constant sectional curvature metric

$$g_\infty = \frac{2}{\sqrt[3]{3}} \left(\eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2 + \eta^3 \otimes \eta^3 \right).$$

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